A SIMPLE PROOF THAT RATIONAL CURVES ON K3 ARE NODAL

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1. Introduction and Statement of Results

The purpose of this paper is to give a simple proof of the following theorem proved in [C2].

Theorem 1.1. *All rational curves in the primitive class of a general K3 surface of genus $g \geq 2$ are nodal.*

Please see [C1] and [C2] for the background of this problem.

We will use a degeneration argument as in [C2]. But instead of degenerating a general K3 surface to a pair of rational surfaces, we will specialize it to a K3 surface S with Picard lattice

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

The Picard group of S is generated by two effective divisors C and F with $C^2 = -2$, $F^2 = 0$ and $C \cdot F = 1$. It can be realized as an elliptic fibration over $\mathbb{P}^1$ with a unique section C, fibers F and $\lambda = 2$. Here $\lambda = c_1(\pi_* \omega)$ is the first Chern class of the Hodge bundle $\pi_* \omega$ of the fibration $\pi : S \to \mathbb{P}^1$ (see [H-M]). It is a standard result that the number of nodal fibers of an elliptic fibration are given by 12λ [H-M, p. 158]. So there are exactly 24 rational nodal curves in the linear series $|F|$ for S general.

This is the same special K3 surface used by Bryan and Leung in their counting of curves on K3 surfaces [B-L]. It is actually the attempt to understand their method that leads us to our proof. We will call a K3 surface with Picard lattice (1.1) a *BL K3* surface.

A BL K3 surface S lies on the boundary of the moduli space of K3 surfaces of genus g with $C + gF$ as the corresponding primitive divisor. Every curve in the linear series $|\mathcal{O}_S(C + gF)|$ is "totally reducible", i.e., it consists of the -2 curve C and g elliptic "tails" attached to C. A curve $D \in |\mathcal{O}_S(C + gF)|$ is the image of a stable rational map only if $D = C \cup m_1 F_1 \cup m_2 F_2 \cup ... \cup m_{24} F_{24}$, where $F_1, F_2, ..., F_{24}$ are 24

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rational nodal curves in the pencil $|F|$ and $\sum_{i=1}^{24} m_i = g$; D is obviously nodal if $m_i \le 1$ for all i. The main problem is, of course, m_i might be greater than 1, i.e., D might be nonreduced, in which case we need to show that when S deforms to a general K3 surface S' of genus g and D correspondingly deforms to a rational curve $D' \subset S'$, D' is necessarily nodal.

It is worthwhile to mention that although this proof looks quite different from the one in [C2], all the basic techniques have already been developed there. By choosing a "good" degeneration as the one used by Bryan-Leung, we eliminate a substantial amount of technicality in the previous proof. In addition, this proof also gives a geometric interpretation of Bryan-Leung's work and makes it possible to redo their counting in the frame of classical algebraic geometry, if one chooses so. Indeed, we will recover part of their counting formula in Appendix B.

We will work exclusively over $\mathbb{C}$ throughout the paper. We use the usual topology instead of Zariski topology most of the time. When we say "neighborhood" of a point or a subscheme, we usually mean analytic neighborhood.

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2. Degeneration of K3 surfaces

Let X be a smooth family of K3 surfaces of genus g over the disk Δ whose central fiber $X_0 = S$ is a BL K3. Let $Y \subset X$ be a flat family of rational curves with $Y_t \subset X_t$ and $Y_0 \in |C + gF|$, where Δ is parameterized by t and Y_t and X_t are general fibers of Y and X over $t \neq 0$. Notice that a base change might be needed to ensure the existence of Y. Let E be one of the 24 rational curves $F_1, F_2, ..., F_{24}$ and $p \in E$ be the node of E. Suppose that Y_0 contains E with multiplicity m. It suffices to show that Y_t has m nodes in the neighborhood of E. If $m = 1$, there is nothing to prove; otherwise, we need to apply the stable reduction to Y by blowing up X and Y along E.

Let $N_{A/B}$ denote the normal bundle of $A \subset B$. Here the normal bundle is defined as the dual of conormal bundle, i.e.,

$$N_{A/B} = \mathcal{H}om(I_A/I_A^2, \mathcal{O}_A), \tag{2.1}$$

where I_A is the ideal sheaf of A in B.

If we blow up X along E (see Figure 1), the exceptional divisor is a ruled surface over E given by $\mathbb{P}N_{E/X}$. We have the exact sequence

$$0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \longrightarrow N_{S/X}\big|_E \longrightarrow 0. \tag{2.2}$$

Notice that

$$N_{E/S} = N_{S/X}\big|_E = \mathcal{O}_E \text{ and } \operatorname{Ext}(\mathcal{O}_E, \mathcal{O}_E) = H^1(\mathcal{O}_E) = \mathbb{C}. \tag{2.3}$$

So (2.2) might not split. Actually this is always the case as long as X is general enough. We claim that

Proposition 2.1. *The exact sequence* (2.2) *does not split provided that the Kodaira-Spencer class of X is general.*

Remark 2.2. Some explanations might be needed on what exactly we mean by a general Kodaira-Spencer class as stated in the above proposition. The first order deformations of S are classified by $H^1(T_S)$ and the Kodaira-Spencer map of X is

$$\mathrm{ks} : T_{\Delta,0} \cong H^0(N_{S/X}) \to H^1(T_S), \tag{2.4}$$

where $T_{\Delta,0}$ is the tangent space of Δ at the origin and T_S is the tangent bundle of S. The versal deformation space of S as a complex manifold has dimension $h^1(T_S) = 20$. However, not every vector in $H^1(T_S)$ is the Kodaira-Spencer class of a projective family X. The algebraic deformations of S are actually given by the vectors of $H^1(T_S)$ lying in a union of countably many subspaces of codimension 1. This is a well-known fact. However, we need the following more precise statement.

Lemma 2.3. *Let X be a smooth family of complex surfaces over Δ whose central fiber $X_0 = S$ is a surface with trivial canonical bundle. Let $Y \subset X$ be a closed subscheme of X of codimension 1 which is flat over Δ and whose central fiber $Y_0 = D$ is an ample divisor on S. Then the Kodaira-Spencer class $\mathrm{ks}(\partial/\partial t)$ of X lies in the subspace $V \subset H^1(T_S)$ consisting of the vectors which are perpendicular to the first Chern class $c_1(D) \in H^1(\Omega_S)$ of the divisor D, i.e.,*

$$\mathrm{ks}(\partial/\partial t) \in V = \{v \in H^1(T_S) : <v, c_1(D)> = 0\}, \tag{2.5}$$

where Ω_S is the cotangent sheaf of S and the pairing $<\cdot,\cdot>$ is given by Serre duality $H^1(T_S) \times H^1(\Omega_S) \to \mathbb{C}$.

On the other hand, if we fix a K3 surface S and an ample divisor D on S, then for each $v \in V$, there exists a pair (X, Y) such that $Y \subset X$, $X_0 = S$, $Y_0 = D$ and the Kodaira-Spencer class of X is v.

We are quite certain that the above lemma is also well known. But since we are unable to locate a reference for it, we will give a proof in Appendix A.

Roughly, Lemma 2.3 says that a general deformation of a surface S with trivial canonical bundle does not preserve any ample divisor D on S. As a direct consequence, we see that a general deformation of an algebraic K3 or abelian surface is no longer algebraic.

Back to our situation and we see that the Kodaira-Spencer class of X lies the subspace of $H^1(T_S)$ perpendicular to $c_1(C+gF)$, i.e.,

$$\mathrm{ks}(\partial/\partial t) \in V = \{v \in H^1(T_S) : <v, c_1(C+gF)> = 0\} \tag{2.6}$$

by Lemma 2.3. Furthermore, for each $v \in V$, there exists a family X whose Kodaira-Spencer class is given by v. In Proposition 2.1, by $\mathrm{ks}(\partial/\partial t)$ being general, we mean that $\mathrm{ks}(\partial/\partial t)$ is general in V.

Proof of Proposition 2.1. The sequence (2.2) splits if and only if the induced map

$$H^0(N_{S/X}\big|_E) \to H^1(N_{E/S}) \tag{2.7}$$

is zero. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_S|_E & \longrightarrow & T_X|_E & \longrightarrow & N_{S/X}|_E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_{E/S} & \longrightarrow & N_{E/X} & \longrightarrow & N_{S/X}|_E & \longrightarrow & 0 \end{array} \tag{2.8}$$

and we can naturally identify $H^0(N_{S/X}\big|_E)$ with $T_{\Delta,0}$. Therefore, the map (2.7) factors through the Kodaira-Spencer map $T_{\Delta,0} \to H^1(T_S)$, the restriction $H^1(T_S) \to H^1(T_S|_E)$ and the surjection

$$H^1(T_S|_E) \to H^1(N_{E/S}) \to H^2(T_E) = 0. \tag{2.9}$$

In short, we have

$$H^0(N_{S/X}\big|_E) \cong T_{\Delta,0} \xrightarrow{\mathrm{ks}} H^1(T_S) \to H^1(T_S|_E) \to H^1(N_{E/S}). \tag{2.10}$$

The last map $H^1(T_S|_E) \to H^1(N_{E/S})$ is actually an isomorphism by the following argument.

By the standard exact sequence

$$0 \to N_{E/S}^\vee \to \Omega_S|_E \to \Omega_E \to 0, \tag{2.11}$$

we have the exact sequence

$$H^0(N_{E/S}) \to \mathrm{Ext}(\Omega_E, \mathcal{O}_E) \to H^1(T_S|_E) \to H^1(N_{E/S}) \to 0. \tag{2.12}$$

Notice that $H^0(N_{E/S}) = \mathbb{C}$ classifies the embedded deformations of $E \subset S$ and $\mathrm{Ext}(\Omega_E, \mathcal{O}_E) = \mathbb{C}$ classifies the versal deformations of E. To show that $H^0(N_{E/S})$ maps nontrivially to $\mathrm{Ext}(\Omega_E, \mathcal{O}_E)$, it suffices to

show that as E varies in the pencil $|\mathcal{O}_S(E)|$, the corresponding Kodaira-Spencer map to the tangent space of the versal deformation space of E at the origin is nontrivial, or equivalently, the map to the versal deformation space of E is unramified over the origin. To see this has to be true, we only need to localize the problem at the node p of E: if the map to the versal deformation space is ramified over the origin, then S is locally given by $xy = t^\alpha$ at p for some $\alpha > 1$; however, this is impossible since S is smooth at p. Therefore, the map

$$H^0(N_{E/S}) \to \operatorname{Ext}(\Omega_E, \mathcal{O}_E) \tag{2.13}$$

is nonzero and hence must be an isomorphism. Thus we conclude that

$$H^1(T_S|_E) \xrightarrow{\sim} H^1(N_{E/S}) = \mathbb{C} \tag{2.14}$$

is an isomorphism.

We have the exact sequence

$$H^1(T_S(-E)) \xrightarrow{f} H^1(T_S) \to H^1(T_S|_E) \cong H^1(N_{E/S}) = \mathbb{C}. \tag{2.15}$$

Combining (2.10) and (2.15), we are left to show that the image of the map $f : H^1(T_S(-E)) \to H^1(T_S)$ does not contain $V \subset H^1(T_S)$ as in (2.6). We claim that that the image of $f : H^1(T_S(-E)) \to H^1(T_S)$ is contained in the subspace W of $H^1(T_S)$ perpendicular to $c_1(E)$, i.e.,

$$\operatorname{Im} f \subset W = \{v \in H^1(T_S) : <v, c_1(E)> = 0\}. \tag{2.16}$$

By Kodaira-Serre duality, we have the following commutative diagram:

$$\begin{array}{ccccc} H^1(T_S(-E)) & \xrightarrow{\sim} & H^1(\Omega_S(E))^\vee & \xrightarrow{\sim} & H^1(\Omega_S(-E)) \\ \downarrow f & & \downarrow & & \downarrow g \\ H^1(T_S) & \xrightarrow{\sim} & H^1(\Omega_S)^\vee & \xrightarrow{\sim} & H^1(\Omega_S). \end{array} \tag{2.17}$$

So we may identify the map f with $g : H^1(\Omega_S(-E)) \to H^1(\Omega_S)$, which is the same as

$$g : H^{1,1}(\mathcal{O}_S(-E)) \to H^{1,1}(\mathcal{O}_S) \tag{2.18}$$

on the Dolbeault cohomologies. For any $\psi \in H^{1,1}(\mathcal{O}_S(-E))$, we have

$$\int_S g(\psi) \wedge c_1(E) = \int_E g(\psi) = 0. \tag{2.19}$$

So (2.16) follows.

On the other hand, we have (2.6). It is trivial that $c_1(C + gF)$ and $c_1(E) = c_1(F)$ are linearly independent in $H^1(\Omega_S)$. So $W \not\supset V$ and a general Kodaira-Spencer class $\operatorname{ks}(\partial/\partial t) \in V$ does not lie in W and hence $\operatorname{ks}(\partial/\partial t) \notin \operatorname{Im} f$. Therefore, $\operatorname{ks}(\partial/\partial t)$ maps nontrivially to

$H^1(T_S|_E) \cong H^1(N_{E/S})$. Consequently, the map (2.7) is not zero and the sequence (2.2) does not split. $\square$

Definition 2.4. There are two ruled surfaces $\mathbb{P}W$ over E, where W is a rank two vector bundle over E satisfying the exact sequence

$$0 \to \mathcal{O}_E \to W \to \mathcal{O}_E \to 0. \tag{2.20}$$

The proof of this fact is not hard, it goes exactly as the classification of the ruled surfaces over an elliptic curve with $e = 0$ (see e.g. [Ha, V, Theorem 2.15]) and we will later give a more geometrical proof of this fact in 3.1. If $W = \mathcal{O}_E \oplus \mathcal{O}_E$, we call $\mathbb{P}W \cong \mathbb{P}^1 \times E$ *trivial*; otherwise if W is indecomposable, we call $\mathbb{P}W$ *twisted*.

Even if the family X we start with is general, we cannot draw the conclusion that $N_{E/X}$ is indecomposable by Proposition 2.1 yet. The problem is that we have already applied a base change to X to ensure the existence of Y. If the degree α of the base change is greater than 1, the Kodaira-Spencer class of the resulting family X will vanish; and if we blow up X along E, the exceptional divisor is simply the trivial ruled surface over E. But eventually a twisted ruled surface over E will show up if we keep blowing up X along E. We explain precisely what we mean by this in the next paragraph.

Let $X^{(1)}$ be the blowup of X along $E_0 = E$ (see Figure 1). The central fiber $X_0^{(1)} = S_0 \cup S_1$ consists of the proper transform S_0 of S and a ruled surface S_1 over E_0. If S_1 is twisted, we stop at $X^{(1)}$. Otherwise, $S_1 \cong \mathbb{P}^1 \times E_0$ is trivial. Notice that the total family $X^{(1)}$ acquires a singularity during the blowup; it has a rational double point $p_1 \neq p_0 \in F_{p_0} \subset S_1$ over the node $p_0 = p$ of E_0, where F_{p_0} is the fiber of $S_1 \to E_0$ over p_0.

Let E_1 be the curve in the pencil $|\mathcal{O}_{S_1}(E_0)|$ passing through p_1. We blow up $X^{(1)}$ along E_1 to obtain $X^{(2)}$. Now the central fiber $X_0^{(2)} = S_0 \cup S_1 \cup S_2$ contains another ruled surface S_2. Notice that we still have the exact sequence

$$0 \to \mathcal{O}_{E_1} \to N_{E_1/X^{(1)}} \to \mathcal{O}_{E_1} \to 0 \tag{2.21}$$

and hence S_2 is one of two ruled surfaces over $E_1 \cong E$ given in Definition 2.4; this is actually true throughout our construction. If S_2 is twisted, we stop at $X^{(2)}$. Otherwise, we do the same thing to $X^{(2)}$ as we did to $X^{(1)}$. Let $F_{p_1} \subset S_2$ be the fiber of $S_2 \to E_1$ over p_1. Notice that $X^{(2)}$ is now singular along F_{p_1}, it is locally given by the equation $xy = t^2$ at a general point of F_{p_1} and there is a point $p_2 \neq p_1 \in F_{p_1}$ where $X^{(2)}$ is locally given by $xy = t^2 z$. Following the convention in [C2], we will slightly abuse the terminology to call a singularity of the

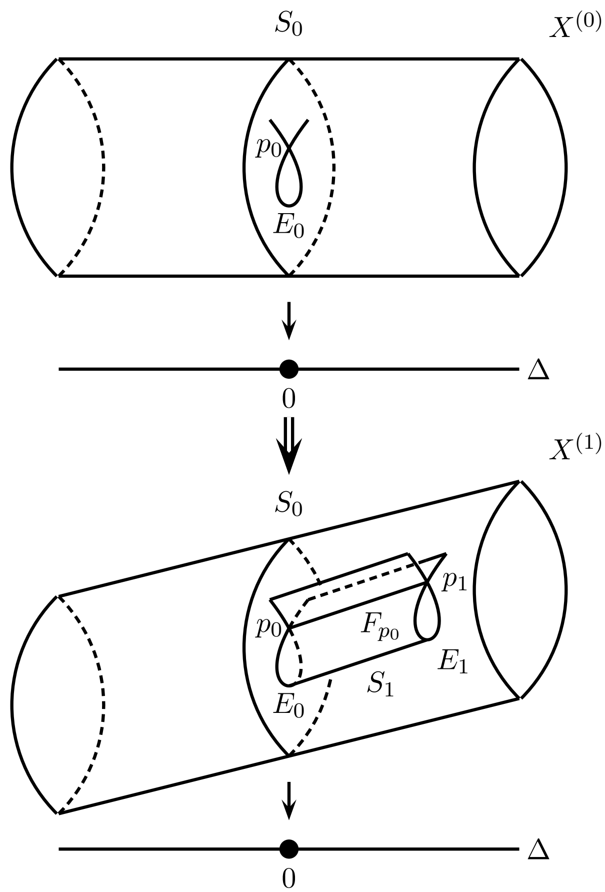

FIGURE 1. The blowup of $X^{(0)} = X$ along $E_0 = E$

type $xy = t^n z$ $(n > 0)$ a rational double point. Let E_2 be the curve in the pencil $|\mathcal{O}_{S_2}(E_1)|$ passing through the rational double point p_2 and a further blowup of $X^{(2)}$ along E_2 will yield $X^{(3)}$. We can continue this process and obtain a blowup sequence

$$... \to X^{(n)} \to X^{(n-1)} \to ... \to X^{(1)} \to X^{(0)} = X \tag{2.22}$$

where $X_0^{(n)} = S_0 \cup S_1 \cup ... \cup S_n$, $S_i \cap S_{i+1} = E_i$, $E_i \cong E$, $E_i \cdot E_{i+1} = 0$ and $S_k \cong \mathbb{P}^1 \times E_{k-1}$ for $1 \leq k \leq n-1$. Let $F_{p_{n-1}}$ be the fiber of $S_n \to E_{n-1}$ over p_{n-1}. Figure 2 shows what happens on the central fiber.

Maybe a better way to understand the singularities of the blowups is to work out the local analytic equations of $X^{(n)}$ over p.

Lemma 2.5. *Let* (2.22) *be the blowup sequence constructed as above. Then for each* $n \geq 1$, $X^{(n)}$ *is singular along* $F_{p_1} \cup F_{p_2} \cup ... \cup F_{p_{n-1}} \cup \{p_n\}$. *At a point* $b \in F_{p_k}$ *and* $b \neq p_k, p_{k+1}$ *for* $0 \leq k \leq n-1$, $X^{(n)}$ *is locally*

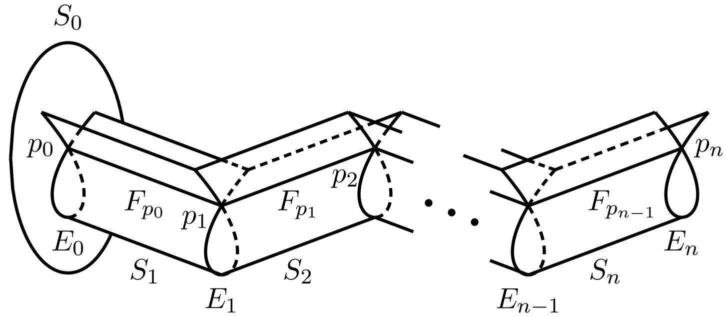

FIGURE 2. The blowup sequence

given by

$$xy = t^{k+1}. \tag{2.23}$$

Locally at p_k *for* $0 \le k \le n-1$,

$$X^{(n)} \cong \Delta^5_{xyzwt}/(xy = t^k z, zw = t) \tag{2.24}$$

and at p_n,

$$X^{(n)} \cong \Delta^4_{xyzt}/(xy = t^n z), \tag{2.25}$$

where Δ^5_{xyzwt} *and* Δ^4_{xyzt} *are the polydisks parameterized by* (x, y, z, w, t) *and* (x, y, z, t), *respectively.*

Proof. We start with $X = X^{(0)}$ which is smooth at $p = p_0$. Choose local coordinates such that $E = E_0$ is cut out by $xy = t = 0$ at p. Blow up $X^{(0)}$ along E_0 and we obtain that

$$X^{(1)} \cong \Delta^3_{xyz_0t}/(xy = tz_0), \tag{2.26}$$

where z_0 is the affine coordinate of $F_{p_0} \cong \mathbb{P}^1$ such that $p_1 \in F_{p_0}$ is given by $z_0 = 0$ and p_0 is given by $z_0 = \infty$. We see from (2.26) that $X^{(1)}$ has a rational double point at p_1. At a point $b \in F_{p_0}$ and $b \ne p_0, p_1$, i.e., for $z_0 \ne 0, \infty$, $X^{(1)}$ is analytically equivalent to (2.23) for $k = 0$. At p_0, i.e., at $z_0 = \infty$, $X^{(1)}$ is given by

$$xyw_0 = t \tag{2.27}$$

where $w_0 = 1/z_0$; this is equivalent to (2.24) for $k = 0$.

Notice that E_1 is cut out by $z_0 = t = 0$. Blow up $X^{(1)}$ along E_1 and we obtain that

$$X^{(2)} \cong \Delta^3_{xyz_1t}/(xy = t^2 z_1), \tag{2.28}$$

where $z_1 = z_0/t$ is the affine coordinate of F_{p_1} such that $p_2 \in F_{p_1}$ is given by $z_1 = 0$ and p_1 is given by $z_1 = \infty$. Obviously, $X^{(2)}$ is given by (2.28) at p_2. At a point $b \in F_{p_1}$ and $b \ne p_1, p_2$, i.e., for $z_1 \ne 0, \infty$, $X^{(2)}$

is analytically equivalent to (2.23) for $k = 1$. At p_1, i.e., at $z_1 = \infty$, $X^{(2)}$ is given by

$$xy = tz_0 \text{ and } w_1 z_0 = t \tag{2.29}$$

where $w_1 = 1/z_1$; this is equivalent to (2.24) for $k = 1$.

Apply this argument inductively for n and we are done. □

As we will see later, the rational double point p_n of $X^{(n)}$ will play an important role in our argument.

The sequence ends at $X^{(n)}$ if $S_n \not\cong \mathbb{P}^1 \times E_{n-1}$ is twisted. Otherwise, let E_n be the curve in $|\mathcal{O}_{S_n}(E_{n-1})|$ passing through p_n and we continue to blow up $X^{(n)}$ along E_n.

Suppose that X is obtained from a family of K3 surfaces with a general (and hence nonvanishing) Kodaira-Spencer class by a base change of degree α. We claim that the above sequence will eventually end and it will end right at $X^{(\alpha)}$. Namely, the blowup sequence will end up as

$$X^{(\alpha)} \to X^{(\alpha-1)} \to \dots \to X^{(1)} \to X^{(0)} = X \tag{2.30}$$

where the corresponding $S_\alpha \subset X_0^{(\alpha)}$ is twisted. This is clear if we reverse the process of base change and blowups. That is, if we blow up X along E before we make a base change, we will obtain $S_\alpha = \mathbb{P}N_{E/X}$ as the exceptional divisor on the central fiber with indecomposable normal bundle $N_{E/X}$ by Proposition 2.1. If we make a base change of degree α afterwards, the total family $\widetilde{X}$ will become singular along E: at a smooth point of E, $\widetilde{X}$ is locally given by the equation $xy = t^\alpha$. We may resolve the generic singularities of $\widetilde{X}$ along E in the same way as in [G-H, Appendix C, p. 39] and we will obtain a chain of ruled surfaces $S_1, S_2, ..., S_{\alpha-1}$ between $S_0 = S$ and S_α. The resulting family is exactly $X^{(\alpha)}$ in (2.30) with the required properties.

For each $1 \le n \le \alpha$, let $Y^{(n)}$ be the proper transform of $Y = Y^{(0)}$ under the map $X^{(n)} \to X$. Depending on our choice of α, the central fiber $Y_0^{(n)}$ could be very "bad"; for example, $Y_0^{(n)}$ could contain one or more of the double curves E_i for $1 \le i \le n-1$. However, we will show that it is possible to choose a suitable α such that the central fiber $Y_0^{(n)}$ of $Y^{(n)}$ is reasonably "well-behaved". Most important of all, we want to make sure that $E_i \not\subset Y_0^{(n)}$.

Actually, the following general statement is true, as a consequence of the stable reduction theorem [KKMS].

Theorem 2.6. *Let X be a flat family of schemes over Δ whose general fibers are smooth and let $Y \subset X$ be a closed subscheme of X of codimension 1 which is flat over Δ. Then there exists a base change of*

X followed by a series of blowups with resulting family $\widetilde{X}$ such that the proper transform $\widetilde{Y}$ of Y meets the singular locus of $\widetilde{X}_0$ properly.

If $\dim X = 2$, one may think of (X, Y) as a family of curves with marked points; it is well known that after a suitable semi-stable reduction $\widetilde{X} \to X$, $\widetilde{Y}$ extends to the sections of $\widetilde{X} \to \Delta$ and the marked points $\widetilde{Y}_0$ can be kept away from the singular locus of $\widetilde{X}_0$. The above theorem is the higher-dimensional analogue, which is not any harder to prove in principle. However, we do not really need Theorem 2.6 since it does not give us any control of $\widetilde{X}$ and hence cannot be applied to our situation directly. Instead, we need the following more precise statement.

Proposition 2.7. *Let X be a smooth family of K3 surfaces over the disk Δ whose central fiber $X_0 = S$ is a BL K3 surface. Suppose that X is obtained from a family of K3 surfaces with a general Kodaira-Spencer class by a base change of degree α. Let $Y \subset X$ be a flat family of rational curves with $Y_0 \in |\mathcal{O}_S(C + gF)|$ and let E be one of the 24 nodal curves in $|\mathcal{O}_S(F)|$ and m be the multiplicity of $E \subset Y_0$.*

Let (2.30) be the blowup sequence constructed as above. Correspondingly, for each $0 \le n \le \alpha$, let $S_n, E_n, p_n, F_{p_n}, Y^{(n)}$ be defined as above.

Let $q_0 = C \cap E_0$ be the intersection between C and E_0 on S_0 and let $F_{q_0} \subset S_1$ be the fiber of $S_1 \to E_0$ over q_0; q_i and F_{q_i} are recursively given by letting $q_i = F_{q_{i-1}} \cap E_i$ and $F_{q_i} \subset S_{i+1}$ be the fiber of $S_{i+1} \to E_i$ over q_i.

There exists a suitable choice of α such that the following holds for each $0 \le n \le \alpha$:

1. *the central fiber $Y_0^{(n)}$ of $Y^{(n)}$ does not contain E_i for $0 \le i \le n-1$;*
2. *$Y^{(n)} \cap S_i$ is a curve in the linear series*

$$\mathbb{P}H^0\left(\mathcal{O}_{S_i}(m_i E_{i-1} + F_{q_{i-1}})\right) \tag{2.31}$$

   for $1 \le i \le n-1$, where $m_1, m_2, ..., m_\alpha$ are α nonnegative integers satisfying $\sum_{i=1}^{\alpha} m_i = m$;

3. *$Y^{(n)} \cap S_n = D \cup \mu E_n$, where D is a curve in the linear series*

$$\mathbb{P}H^0\left(\mathcal{O}_{S_n}(m_n E_{n-1} + F_{q_{n-1}})\right) \tag{2.32}$$

   and $\mu = \sum_{i=n+1}^{\alpha} m_i$;

4. *$F_{q_i} \subset \left(Y^{(n)} \cap S_i\right)$ for $1 \le i \le n \le \alpha - 1$.*

Although the general results on stable reduction such as Theorem 2.6 cannot be applied to Proposition 2.7 directly, its proof is actually carried out by explicitly applying semi-stable reduction to $X^{(\alpha)}$.

By Proposition 2.7, $Y_0^{(\alpha)}$ looks as follows: the components of $Y_0^{(\alpha)}$ over E consist of

$$(F_{q_0} \cup D_1) \cup (F_{q_1} \cup D_2) \cup ... \cup (F_{q_{\alpha-2}} \cup D_{\alpha-1}) \cup \Gamma, \tag{2.33}$$

where $D_i \subset S_i$, $D_i \in |\mathcal{O}_{S_i}(m_i E_{i-1})|$, $E_i \not\subset D_i$ for $1 \le i \le \alpha - 1$, $\Gamma \subset S_\alpha$ and $\Gamma \in |\mathcal{O}_{S_\alpha}(m_\alpha E_{\alpha-1} + F_{q_{\alpha-1}})|$. We will call the components D_i "wandering components". Actually, we have

Proposition 2.8. *With all the notations as above, then*

$$m_1 = m_2 = ... = m_{\alpha-1} = 0, \tag{2.34}$$

i.e., $D_i = \emptyset$ for $1 \le i \le \alpha - 1$ and there are no wandering components at all.

Therefore, "interesting" things only happen on the twisted ruled surface S_α. Among the components $F_{q_0} \cup F_{q_1} \cup ... \cup F_{q_{\alpha-2}} \cup \Gamma$ of $Y_0^{(\alpha)}$, F_{q_i}'s are a chain of rational curves connecting C and Γ and they will be contracted under stable reduction; the only nontrivial part is $\Gamma \subset S_\alpha$ which maps to E with a degree m map. One of main steps of our proof is to classify all possible configurations of Γ.

Let $\delta(A)$ denote the total δ-invariant of the singularities of a curve A and let $\delta(A, B)$ denote the total δ-invariant of the singularities of A in the (analytic) neighborhood of B. The latter notation $\delta(A, B)$ is used under two circumstances:

1. if $B \subset A$ is a closed subscheme of A, $\delta(A, B)$ is simply the total δ-invariant of the singularities p of A with $p \in B$;
2. if Υ is a family of curves over the disk Δ, $A = \Upsilon_t$ is the general fiber of $\Upsilon \to \Delta$ and $B \subset \Upsilon_0$ is a closed subscheme of the central fiber Υ_0, then $\delta(\Upsilon_t, B)$ is the total δ-invariant of the singularities of Υ_t in the neighborhood of B; notice that this is well-defined.

We claim that

Proposition 2.9. *Suppose that Proposition 2.7 and 2.8 are true. With all the notations as above,*

$$\delta(Y_t^{(\alpha)}, \Gamma) \ge m \tag{2.35}$$

and if the equality holds, the general fiber $Y_t^{(\alpha)}$ of $Y^{(\alpha)}$ has exactly m nodes in the neighborhood of Γ. Or equivalently, $\delta(Y_t, E) \ge m$ and if the equality holds, the general fiber Y_t of Y has exactly m nodes in the neighborhood of E.

Notice that the total δ-invariant of Y_t is g and

$$g = \delta(Y_t) \ge \sum \delta(Y_t, E) \tag{2.36}$$

where we sum over all the 24 nodal fibers E of $S \to \mathbb{P}^1$. By Proposition 2.9, the RHS of (2.36) is at least the sum of the multiplicities of E in Y_0, which is g. So we must have $\delta(Y_t, E) = m_E$ for each nodal fiber E, where m_E is the multiplicity of E in Y_0. By Proposition 2.9 again, Y_t is nodal in the neighborhood of each E. And our main theorem follows.

The rest of the paper is organized as follows. In Sec. 3, we will introduce some preliminary results that will be needed later in our proof, which include a geometrical construction of the twisted ruled surface over E and some local results on the deformation of curve singularities. Next we will prove Proposition 2.9 in Sec. 4, during which we will give a classification for all possible configurations of Γ and the stable reduction over it. The proofs of Proposition 2.7 and 2.8 will be postponed until Sec. 5.

3. Preliminaries

3.1. **Construction of the Twisted Ruled Surface.** Let E be a rational curve with one node and let W be a rank 2 vector bundle over E satisfying the exact sequence (2.20). As mentioned before, there are two isomorphism classes of $\mathbb{P}W$: one is "trivial" and the other is "twisted". We will give an explicit geometric construction of the latter.

Let $\nu : \widetilde{E} \to E$ be the normalization of E. Since $\widetilde{E} \cong \mathbb{P}^1$, $\nu^*(W)$ splits to $\mathcal{O}_{\widetilde{E}} \oplus \mathcal{O}_{\widetilde{E}}$ on $\widetilde{E}$. And this induces a map $\nu : \mathbb{P}^1 \times \widetilde{E} \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}W$, which is just the normalization of $\mathbb{P}W$. Intuitively, we say ν "unfolds" $\mathbb{P}W$.

We use E and $\widetilde{E}$ to denote the zero sections of $\mathbb{P}W \to E$ and its normalization $\mathbb{P}^1 \times \widetilde{E} \to \widetilde{E}$, respectively.

Let $a, b \in \widetilde{E}$ be the preimages of the node $p \in E$. Let F_a, F_b be the fibers of $\mathbb{P}^1 \times \widetilde{E}$ over a, b and let F_p be the fiber of $\mathbb{P}W \to E$ over p. One can think of $\mathbb{P}W$ being constructed from $\mathbb{P}^1 \times \widetilde{E}$ by "gluing" two fibers F_a and F_b.

Let $\nu_a : F_a \to F_p$ and $\nu_b : F_b \to F_p$ be the maps induced by ν. We have a natural identification ϕ_{ab} between F_a and F_b on $\mathbb{P}^1 \times \widetilde{E}$, which simply sends $x \in F_a$ to $y \in F_b$ if there is a curve in the pencil $|\widetilde{E}|$ passing through x and y. So $h = \phi_{ba} \circ \nu_b^{-1} \circ \nu_a$ is an automorphism of $F_a \cong \mathbb{P}^1$, where $\phi_{ba} = \phi_{ab}^{-1}$.

If $x \in F_a$ is a fixed point of h, i.e., $h(x) = x$, the curve $D \in |\widetilde{E}|$ passing through x and $\phi_{ab}(x)$ maps to a curve $\nu(D) \in |E|$. If W is indecomposable, there is only one curve in $|E|$. So h can have only one fixed point. If we represent h by a matrix $H \in GL(2)$, H has only one

eigenvector and is hence equivalent to

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

In fact, λ in (3.1) classifies all the extensions in $\mathrm{Ext}(\mathcal{O}_E, \mathcal{O}_E) = \mathbb{C}$. For $\lambda = 0$, we obtain $\mathbb{P}^1 \times E$; for $\lambda \neq 0$, we obtain $\mathbb{P}W$ with W indecomposable and they are isomorphic to each other (see Figure 3).

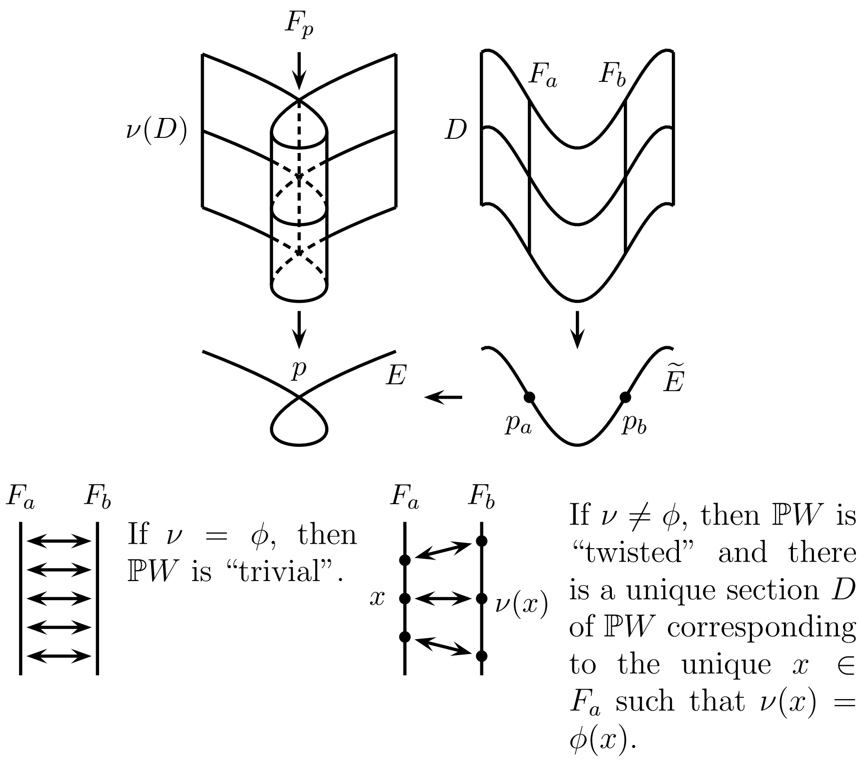

Figure 3. "trivial" vs "twisted"

Remark 3.1. On an interesting though unrelated issue, one may ask what kind of surfaces we get if we glue $\mathbb{P}^1 \times \mathbb{P}^1$ along F_a and F_b via an automorphism h which has two fixed points, i.e., whose corresponding matrix representation H has two eigenvectors. The resulting surface S will have exactly two sections D_1 and D_2 with self-intersection zero. So what kind of surface is S? Actually, $S = \mathbb{P}(\mathcal{O}_E \oplus L_E)$ where L_E is a nontrivial line bundle on E with $\deg L_E = 0$. The two sections D_1 and D_2 are not linearly equivalent on S and they correspond to the global sections of $\mathcal{O}_E \oplus L_E$ and $\mathcal{O}_E \oplus L_E^{-1}$, respectively. I would like to thank James McKernan for pointing this out to me.

3.2. **A Key Lemma.** This is basically Lemma 2.2 in [C1] or Lemma 2.1 in [C2].

Lemma 3.2. *Let $X \subset \Delta^3_{xyz} \times \Delta_t$ be a family of surfaces given by $xy = t^\alpha$ for some $\alpha > 0$. Let X_0 be the central fiber of X over Δ_t and $X_0 = R_1 \cup R_2$ where $R_1 = \{x = t = 0\}$ and $R_2 = \{y = t = 0\}$ and let $E = R_1 \cap R_2$. Let Y be a flat family of curves over Δ_t and $\pi : Y \to X$ be a proper morphism preserving the base Δ_t. Suppose that $E \not\subset \pi(Y_0)$, where Y_0 is the central fiber of Y. Let $Y_0 = \Gamma_1 \cup \Gamma_2$ with $\pi(\Gamma_1) \subset R_1$ and $\pi(\Gamma_2) \subset R_2$. Then $\pi(\Gamma_1) \cdot E = \pi(\Gamma_2) \cdot E$, where the intersections $\pi(\Gamma_1) \cdot E$ and $\pi(\Gamma_2) \cdot E$ are taken on the surfaces R_1 and R_2, respectively.*

The proof of this lemma is not hard. The readers may find a proof in [C1] or [C2].

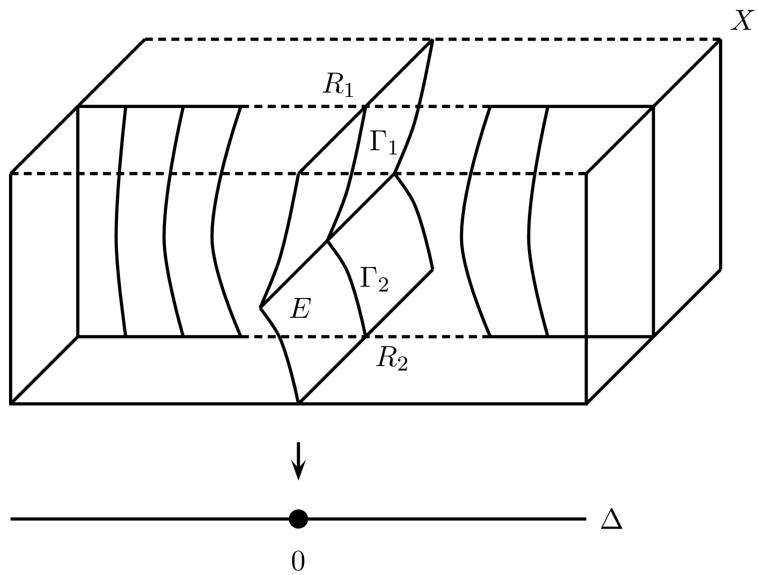

FIGURE 4. Lemma 3.2

Definition 3.3. Let Y be a one-parameter family of curves over Δ and let $p \in Y_0$ be a point on the central fiber Y_0. Even if Y is irreducible globally, it is still possible that Y is reducible in an analytic neighborhood of p. That is, if we let U be an analytic neighborhood of Y at p, U might be reducible such that $U = \cup V_i$ where we call each V_i a *local irreducible component* of Y at p. This happens if Y is not normal and the general fiber Y_t is singular in the neighborhood of p. If Y breaks into several local irreducible components at p, the normalization of Y will make these components disconnected. Let Γ_1 and Γ_2 be two local

branches of Y_0 at p. We call Γ_1 is *locally separated from* Γ_2 at p if Γ_1 and Γ_2 do not lie on the same local irreducible component of Y at p, or equivalently, Γ_1 and Γ_2 become disconnected on the normalization of Y. And we call Y is *totally separated* at p if any two branches of Y_0 at p are locally separated from each other, i.e., if $Y_0 = \mu_1\Gamma_1 \cup \mu_2\Gamma_2 \cup ... \cup \mu_k\Gamma_k$ at p, Γ_i is locally separated from Γ_j for all $1 \le i \ne j \le k$.

Remark 3.4. It is necessary to point out that Lemma 3.2 is a local result. So it holds for every local irreducible component of Y at $\pi^{-1}(p)$, where $p \in X$ is the origin. For example, suppose that $Y_0 = \Gamma_1 \cup \Gamma_2$ with $\pi(\Gamma_i) \subset R_i$ for $i = 1, 2$ and Γ_1 is reduced and locally irreducible. Then we certainly have $\pi(\Gamma_1) \cdot E = \pi(\Gamma_2) \cdot E$ by the lemma; in particular, this means $\Gamma_2 \ne \emptyset$. In addition, we can also conclude by the lemma that Y is locally irreducible at $\pi^{-1}(p)$, which implies that no component of Γ_2 is locally separated from Γ_1. As for another example, take $Y_0 = \cup_{i=1}^4 \Gamma_i$ with $\pi(\Gamma_1), \pi(\Gamma_2) \subset R_1$ and $\pi(\Gamma_3), \pi(\Gamma_4) \subset R_2$ and suppose that each $\pi(\Gamma_i)$ meets E transversely. Then we may conclude by the lemma that Y consists of at most two local irreducible components and if this happens, we have either Γ_1 and Γ_3 lie on one component and Γ_2 and Γ_4 lie on the other or Γ_1 and Γ_4 lie on one component and Γ_2 and Γ_3 lie on the other; in particular, Y cannot be totally separated at $\pi^{-1}(p)$.

For a three-fold rational double point $p \in X$ given by $xy = t^\alpha z$, we can resolve X at p by blowing up one of the two surfaces of X_0 at p, i.e., let $\widetilde{X} \subset X \times \mathbb{P}^1$ be the resolution given by

$$\frac{x}{z} = \frac{t^\alpha}{y} = \frac{W_1}{W_0}, \tag{3.2}$$

where (W_0, W_1) is the homogeneous coordinate of $\mathbb{P}^1$. Strictly speaking, it is not a resolution of singularities because $\widetilde{X}$ is still singular if $\alpha > 1$. But now $\widetilde{X}$ is given by $wy = t^\alpha$ along its singular locus, where we may apply Lemma 3.2 to obtain the following corollary.

Corollary 3.5. *Let X, R_1, R_2, E, π, Y be defined as in Lemma 3.2 except that X is given by $xy = t^\alpha z$ instead. Suppose that Y_0 contains a component Γ_1 such that $\pi(\Gamma_1) \subset R_1$ is tangent to E at the origin p. Then there must exist a component Γ_2 of Y_0 such that $\pi(\Gamma_2) \subset R_2$ passes through p. In particular, Y cannot be totally separated at point q where $\pi(q) = p$ and $q \in \Gamma_1$.*

Proof. See [C2, Corollary 2.1]. □

3.3. **Some Results on Curve Singularities.** The following lemma is basically a combination of Corollary 4.1 and Proposition 4.3 in [C2].

Lemma 3.6. *Let $Y \subset \Delta^2 \times \Delta_t$ be a reduced flat family of curves over Δ_t with central fiber $Y_0 = \mu_1\Gamma_1 \cup \mu_2\Gamma_2 \cup ... \cup \mu_n\Gamma_n$, where μ_i is the multiplicity of the component Γ_i in Y_0. Suppose that Y is totally separated at the origin p. Then*

$$\delta(Y_t) \geq \sum_{1\leq r<s\leq n} \mu_r\mu_s(\Gamma_r \cdot \Gamma_s) \tag{3.3}$$

where the intersections $\Gamma_r \cdot \Gamma_s$ are taken on $\{t = 0\} \cong \Delta^2$.

If the equality holds in (3.3) and we further assume that

A1. *Γ_r and Γ_s meet transversely, i.e., $\Gamma_r \cdot \Gamma_s = 1$ for $1 \leq r < s \leq n$, and*

A2. *for each irreducible component $Z \subset Y$ of Y, the central fiber Z_0 of Z is reduced, i.e., Y consists of exactly $\sum_{i=1}^n \mu_i$ irreducible components,*

then Y_t is nodal.

Proof. See [C2, Sec. 4]. □

Remark 3.7. Here is an example how to apply Lemma 3.6. Let

$$Y \subset \Delta^2_{xy} \times \Delta_t \tag{3.4}$$

be a reduced flat family of curves whose central fiber Y_0 is given by $x^my^n = 0$, i.e., $Y_0 = m\Gamma_1 \cup n\Gamma_2$ where Γ_1 and Γ_2 are the curves $\{x = t = 0\}$ and $\{y = t = 0\}$, respectively. Suppose that Y is totally separated at the origin p. That is to say that for each irreducible component $Z \subset Y$, either $Z_0 = m'\Gamma_1$ for some $m' \leq m$ or $Z_0 = n'\Gamma_2$ for some $n' \leq n$. Then Lemma 3.6 yields that $\delta(Y_t) \geq mn$. If we further assume that $\delta(Y_t) = mn$ and Y has exactly $m+n$ irreducible components, then Y_t has exactly mn nodes as singularities.

The above lemma can be applied to a family of curves in the neighborhood of a three-fold rational double point $xy = t^\alpha z$.

Corollary 3.8. *Let $X \subset \Delta^3_{xyz} \times \Delta_t$ be a family of surfaces given by $xy = t^\alpha z$ for some $\alpha > 0$ and let R_1, R_2, E be defined as in Lemma 3.2. Let $Y \subset X$ be a reduced closed subscheme of X with codimension 1 and suppose that $E \not\subset Y_0$. Let $Y_0 = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset R_1$ and $\Gamma_2 \subset R_2$. If*

A1. *each irreducible component of Y_0 meets E transversely and*

A2. *Y is totally separated at the origin p,*

then

$$\delta(Y_t) \geq \mu_1\mu_2 \tag{3.5}$$

where $\mu_1 = \Gamma_1 \cdot E$ *and* $\mu_2 = \Gamma_2 \cdot E$. *If the equality holds in* (3.5) *and we further assume that*

A3. *for each irreducible component* $Z \subset Y$ *of* Y, *the central fiber* Z_0 *of* Z *is reduced, i.e.,* Y *consists of exactly* $\mu_1 + \mu_2$ *irreducible components,*

then Y_t *is nodal.*

This is a weak version of Proposition 4.4 and 4.5 in [C2], which can be proved by first resolving X as in (3.2) and then applying Lemma 3.6. Please see [C2, Sec. 4] for the details.

4. Proof of Proposition 2.9

First we "unfold" the twisted ruled surface S_α as in 3.1. Let

$$\nu : \widetilde{S}_\alpha = \mathbb{P}^1 \times \widetilde{E}_{\alpha-1} \to S_\alpha \tag{4.1}$$

be the normalization of S_α, where $\widetilde{E}_{\alpha-1}$ is the normalization of $E_{\alpha-1}$. Let $a, b \in \widetilde{E}_{\alpha-1}$ be the preimages of the node $p_{\alpha-1}$ and let $F_a, F_b \subset \widetilde{S}_\alpha$ be the fibers over a and b. Let $\nu_a : F_a \to F_{p_{\alpha-1}}$ and $\nu_b : F_b \to F_{p_{\alpha-1}}$ be the maps induced by ν and let $\varepsilon_{ab} = \nu_b^{-1} \circ \nu_a$ and $\varepsilon_{ba} = \nu_a^{-1} \circ \nu_b$. We will abbreviate both ε_{ab} and ε_{ba} to ε most of time since it is usually clear which one we are using, i.e., $\varepsilon(u) = \varepsilon_{ab}(u)$ if $u \in F_a$ and $\varepsilon(u) = \varepsilon_{ba}(u)$ if $u \in F_b$. Also we write $u \xrightarrow{\varepsilon} w$ if $w = \varepsilon(u)$.

Let ϕ_{ab} and ϕ_{ba} be defined as in 3.1, i.e., $w = \phi_{ab}(u)$ if $u \in F_a$ and $w \in F_b$ lie on a curve in the pencil $|\mathcal{O}_{\widetilde{S}_\alpha}(\widetilde{E}_{\alpha-1})|$. Again, we will abbreviate both ϕ_{ab} and ϕ_{ba} to ϕ, i.e., $\phi(u) = \phi_{ab}(u)$ if $u \in F_a$ and $\phi(u) = \phi_{ba}(u)$ if $u \in F_b$. We write $u \xrightarrow{\phi} w$ if $w = \phi(u)$. Also we use the notation $\overline{uw}$ to denote the curve in $|\mathcal{O}_{\widetilde{S}_\alpha}(\widetilde{E}_{\alpha-1})|$ passing through u and w if $u \xrightarrow{\phi} w$.

Let $r_a \in F_a$ and $r_b \in F_b$ be the preimages of the rational double point p_α. Using the notations just defined, we have $r_a \xrightarrow{\varepsilon} r_b$ and $r_b \xrightarrow{\varepsilon} r_a$.

Let $\widetilde{\Gamma} = \nu^{-1}(\Gamma) \subset \widetilde{S}_\alpha$. Suppose that $\widetilde{\Gamma}$ meets F_a at a point $u \neq r_a$ with multiplicity k. The branches of $\widetilde{\Gamma}$ at u map to the branches of Γ lying on one of the two surfaces of $X_0^{(\alpha)}$ at $\nu(u)$, where $X^{(\alpha)}$ is locally given by $xy = t^\alpha$. So we can apply Lemma 3.2 to $Y^{(\alpha)} \subset X^{(\alpha)}$ at $\nu(u)$ and conclude that there must be branches of Γ lying on the other surface of $X_0^{(\alpha)}$ at $\nu(u)$ and the branches on both surfaces must meet $F_{p_{\alpha-1}}$ at $\nu(u)$ with the same multiplicity k. Correspondingly, $\widetilde{\Gamma}$ must meet F_b at $w = \varepsilon(u)$ with multiplicity k. Therefore, if $\widetilde{\Gamma}$ meets F_a at $u \neq r_a$ with multiplicity k, $\widetilde{\Gamma}$ must meet F_b at $w = \varepsilon(u)$ with the same

multiplicity k. Similarly, if $\widetilde{\Gamma}$ meets F_b at $w \neq r_b$ with multiplicity k, $\widetilde{\Gamma}$ must meet F_a at $u = \varepsilon(w)$ with the same multiplicity k. So we can pair each $u \neq r_a \in \widetilde{\Gamma} \cap F_a$ with $w = \varepsilon(u) \neq r_b \in \widetilde{\Gamma} \cap F_b$ and $(\widetilde{\Gamma} \cdot F_a)_u = (\widetilde{\Gamma} \cdot F_b)_w$. And for the remaining pair $r_a \xrightarrow{\varepsilon} r_b$, we must have $(\widetilde{\Gamma} \cdot F_a)_{r_a} = (\widetilde{\Gamma} \cdot F_b)_{r_b}$. In summary, we have

$$(\widetilde{\Gamma} \cdot F_a)_u = (\widetilde{\Gamma} \cdot F_b)_w \tag{4.2}$$

for any pair of points $u \in F_a$ and $w \in F_b$ with $u \xrightarrow{\varepsilon} w$.

Let $N \subset \Gamma$ be the irreducible component of Γ with

$$N \in |\mathcal{O}_{S_\alpha}(F_{q_{\alpha-1}} + \mu E_{\alpha-1})| \tag{4.3}$$

for some $\mu \leq m$. And let $\widetilde{N} = \nu^{-1}(N) \subset \widetilde{S}_\alpha$.

Let $\widetilde{Y} \to Y^{(\alpha)}$ be the stable reduction of $Y^{(\alpha)}$ after normalization. Namely, $\widetilde{Y}_t$ is the normalization of $Y_t^{(\alpha)}$ on the general fibers and

$$\widetilde{Y}_0 \to Y_0^{(\alpha)} \tag{4.4}$$

is a stable map on the central fiber. We say a component $M_1 \subset \widetilde{Y}_0$ is joined to another component $M_2 \subset \widetilde{Y}_0$ over a point $s \in Y_0^{(\alpha)}$ if the two components M_1 and M_2 are joined by a chain of curves contracted to s.

Consider the component of $\widetilde{Y}_0$ that dominates N. It must be isomorphic to $\widetilde{N} \subset \widetilde{S}_\alpha$. So we use the same notation $\widetilde{N}$ to denote this component.

We call a sequence of points $\{u_0, w_0, u_1, w_1, ..., u_n, w_n\} \subset \widetilde{\Gamma} \cap (F_a \cup F_b)$ an *S-chain* if $u_0 \in F_a$ and

$$u_0 \xrightarrow{\varepsilon} w_0 \xrightarrow{\phi} u_1 \xrightarrow{\varepsilon} w_1 \xrightarrow{\phi} ... \xrightarrow{\varepsilon} w_{n-1} \xrightarrow{\phi} u_n \xrightarrow{\varepsilon} w_n. \tag{4.5}$$

Notice that $u_{i+1} = h(u_i)$ where $h = \phi \circ \varepsilon \in \mathrm{Aut}(F_a) \cong \mathrm{Aut}(\mathbb{P}^1)$ is the automorphism of $\mathbb{P}^1$ given by (3.1) with $\lambda \neq 0$ if we let $a \in F_a$ be the point at ∞. Obviously, $h^k(u) \neq u$ for any $u \neq a$ and $k \neq 0$ and hence $u_i \neq u_j$ for any $i \neq j$. Similarly, $w_i \neq w_j$ for any $i \neq j$. Therefore, the points in an S-chain are distinct.

An S-chain is maximal if it is not contained in a longer S-chain. We claim that

Proposition 4.1. *A maximal S-chain must contain either r_a or r_b.*

Proof. Let $\{u_0, w_0, u_1, w_1, ..., u_n, w_n\}$ be a maximal S-chain and

$$r_a, r_b \notin \{u_0, w_0, u_1, w_1, ..., u_n, w_n\}. \tag{4.6}$$

Since $\{u_0, w_0, u_1, w_1, ..., u_n, w_n\}$ is maximal, there does not exists $w \in \widetilde{\Gamma} \cap F_b$ such that $w \xrightarrow{\phi} u_0$ and there is no curve $\overline{wu_0} \subset \widetilde{\Gamma}$. So $\widetilde{N}$ has to

pass through u_0. Similarly, there is no point $u \in F_a$ such that $\overline{w_n u} \subset \widetilde{\Gamma}$ and hence $\widetilde{N}$ must pass through w_n.

Applying Lemma 3.2 to the point $\nu(u_0) = \nu(w_0)$, we see that the branch of $\widetilde{N}$ at u_0 is joined to either the branch of $\widetilde{N}$ at w_0 or a component M_1 dominating $\nu(\overline{w_0 u_1})$ over $\nu(u_0)$. If it is the former case that the branch of $\widetilde{N}$ at u_0 is joined to the branch of $\widetilde{N}$ at w_0 over $\nu(u_0)$, it contradicts the fact that the dual graph of $\widetilde{Y}_0$ is a tree. Otherwise, if $\widetilde{N}$ is joined to M_1 over $\nu(u_0)$, we continue to apply Lemma 3.2 to the point $\nu(u_1) = \nu(w_1)$ and see that M_1 is joined to either $\widetilde{N}$ or a component M_2 dominating $\nu(\overline{w_1 u_2})$ over $\nu(u_1)$. If it is the former case, we again get a circuit in the dual graph of $\widetilde{Y}_0$. We may continue this argument and obtain that $\widetilde{N}$ is joined to M_1 over $\nu(u_0)$, M_1 is joined to M_2 over $\nu(u_1)$ and so on; finally, we have M_{n-1} is joined to M_n over $\nu(u_{n-1})$, where $M_n \subset \widetilde{Y}_0$ is a component dominating $\nu(\overline{w_{n-1} u_n})$. As mentioned before, there is no curve $\overline{w_n u} \subset \widetilde{\Gamma}$. So M_n is joined to $\widetilde{N}$ over $\nu(u_n) = \nu(w_n)$. Once again, we obtain a circuit in the dual graph of $\widetilde{Y}_0$. Contradiction.

Figure 5 illustrates our argument. Here $\widetilde{N}$ passes through u_0 and w_i. Then there will be a loop between $\nu(u_0) = \nu(w_0)$ and $\nu(u_i) = \nu(w_i)$ on $\widetilde{Y}_0$ and consequently, $p_a(\widetilde{Y}_0) > 0$. This is a contradiction. □

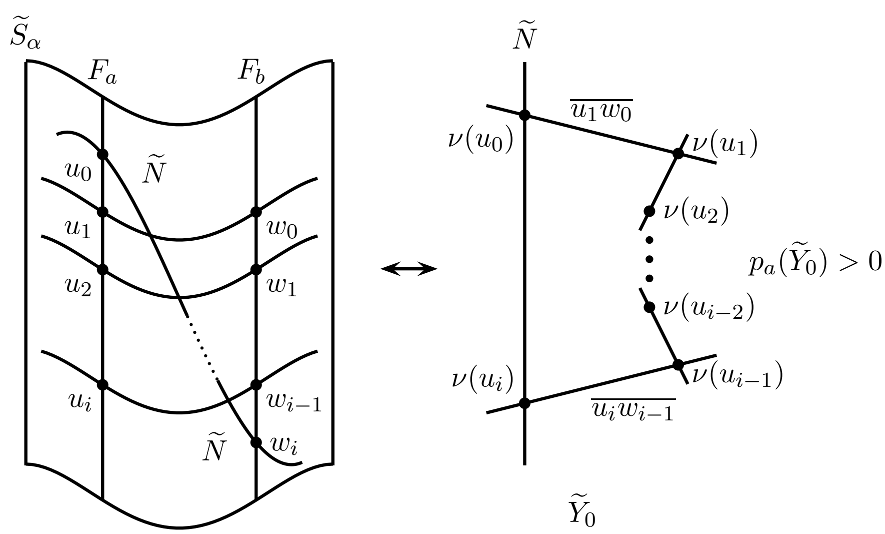

FIGURE 5. Proposition 4.1

The difference between the points r_a, r_b and the other points u_i, w_i lies in that at $\nu(u_i) = \nu(w_i) \neq p_\alpha$, $X^{(\alpha)}$ is locally given by $xy = t^\alpha$ so

Lemma 3.2 applies at $\nu(u_i)$, while $X^{(\alpha)}$ has a rational double point at $p_\alpha = \nu(r_a) = \nu(r_b)$ and hence Lemma 3.2 does not apply at $\nu(r_a)$.

It is obvious that any two maximal S-chains are disjoint from each other. Combining this with Proposition 4.1, we see that there is only one maximal S-chain, i.e., the points in $\widetilde{\Gamma} \cap (F_a \cup F_b)$ form an S-chain in a certain order. We can arrange the points in $\widetilde{\Gamma} \cap (F_a \cup F_b)$ in the following way:

$$\begin{aligned} u_{-k} \xrightarrow{\varepsilon} w_{-k} \xrightarrow{\phi} u_{-k+1} \xrightarrow{\varepsilon} w_{-k+1} \xrightarrow{\phi} \dots \xrightarrow{\phi} u_0 \xrightarrow{\varepsilon} w_0 \\ \xrightarrow{\phi} u_1 \xrightarrow{\varepsilon} w_1 \xrightarrow{\phi} \dots \xrightarrow{\phi} u_l \xrightarrow{\varepsilon} w_l, \end{aligned} \tag{4.7}$$

where $u_0 = r_a$, $w_0 = r_b$ and $k, l \geq 0$.

Proposition 4.2. *Let μ_i be the multiplicity of the curve $\overline{w_i u_{i+1}}$ in $\widetilde{\Gamma}$ for $-k \leq i \leq l-1$. Then*

A1. $\mu_{-k}, \mu_{-k+1}, ..., \mu_0, \mu_1, ..., \mu_{l-1}$ *satisfy*

$$1 \leq \mu_{-k} \leq \mu_{-k+1} \leq ... \leq \mu_{-1} \text{ and } \mu_0 \geq \mu_1 \geq ... \geq \mu_{l-1} \geq 1; \tag{4.8}$$

A2. *$Y^{(\alpha)}$ is totally separated at $p_\alpha = \nu(u_0) = \nu(w_0)$ and hence*

$$|\mu_{-1} - \mu_0| \leq 1; \tag{4.9}$$

A3. *if N meets $\nu(\overline{w_i u_{i+1}})$ at a point $s \neq \nu(w_i), \nu(u_{i+1})$, $Y^{(\alpha)}$ is totally separated at s.*

Proof. By (4.2), we have

$$(\widetilde{\Gamma} \cdot F_a)_{u_i} = (\widetilde{\Gamma} \cdot F_b)_{w_i} \tag{4.10}$$

for $-k \leq i \leq l$. So (4.8) is equivalent to the statement that $\widetilde{N}$ meets F_a only at the points $u_{-k}, u_{-k+1}, ..., u_{-1}, u_0$ and meets F_b only at the points $w_0, w_1, ..., w_{l-1}, w_l$. Obviously, $\widetilde{N}$ must pass through u_{-k} since there is no curve $\overline{w u_{-k}} \subset \widetilde{\Gamma}$. For the same reason, $w_l \in \widetilde{N}$.

Suppose that $w_{-i} \in \widetilde{N}$ for some $1 \leq i \leq k$ and i is the largest number for this to hold. Applying Lemma 3.2 to $\nu(w_{-i}) = \nu(u_{-i})$, we see that $\widetilde{N}$ is joined to a component $M_1 \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_{-i-1} u_{-i}})$ over $\nu(w_{-i})$; continuing applying Lemma 3.2, we see that M_1 is joined to a component M_2 dominating $\nu(\overline{w_{-i-2} u_{-i-1}})$ over $\nu(w_{-i-1})$, M_2 is joined to M_3 dominating $\nu(\overline{w_{-i-3} u_{-i-2}})$ over $\nu(w_{-i-2})$ and so on. Finally, we have M_{k-i} dominating $\nu(\overline{w_{-k} u_{-k+1}})$ is joined to $\widetilde{N}$ over $\nu(w_{-k})$ and we obtain a circuit in the dual graph of $\widetilde{Y}_0$. Contradiction. Therefore, $w_{-k}, w_{-k+1}, ..., w_{-1} \notin \widetilde{N}$. Similarly, $u_1, u_2, ..., u_l \notin \widetilde{N}$.

If $Y^{(\alpha)}$ is not totally separated at p_α, we have three cases

1. a component $M_1 \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_0 u_1})$ is joined to a component $M_2 \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_{-1} u_0})$ over p_α;
2. a component $M_1 \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_0 u_1})$ is joined to $\widetilde{N}$ over p_α;
3. a component $M_2 \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_{-1} u_0})$ is joined to $\widetilde{N}$ over p_α.

In either of these cases, we can argue in the same way as before to show that there is a circuit in the dual graph of $\widetilde{Y}_0$. Therefore, $Y^{(\alpha)}$ is totally separated at p_α. As a consequence, by Corollary 3.5 $\widetilde{N}$ can be neither tangent to F_a at u_0 nor tangent to F_b at w_0. So if $\widetilde{N}$ meets F_a and F_b at u_0 and w_0, it must meet F_a and F_b transversely at these points. Combining this with the fact that $(\widetilde{\Gamma} \cdot F_a)_{u_0} = (\widetilde{\Gamma} \cdot F_b)_{w_0}$, we obtain (4.9).

Finally for (A3), if $Y^{(\alpha)}$ is not totally separated at $s = N \cap \nu(\overline{w_i u_{i+1}})$, then $\widetilde{N}$ will be joined to a component $M \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_i u_{i+1}})$ over s. Again, we may use the same argument as before to show that there is a circuit in the dual graph of $\widetilde{Y}_0$. □

Since $\widetilde{\Gamma} = \left(\cup_{i=-k}^{l-1} \mu_i \overline{w_i u_{i+1}}\right) \cup \widetilde{N}$,

$$\sum_{i=-\infty}^{\infty} \mu_i + \mu = m \tag{4.11}$$

where μ is defined in (4.3) and we let $\mu_i = 0$ if $i < -k$ or $i \geq l$. It follows from (4.10) that

$$\mu_i - \mu_{i-1} = (\widetilde{N} \cdot F_a)_{u_i} \tag{4.12}$$

for $i \leq -1$ and

$$\mu_j - \mu_{j+1} = (\widetilde{N} \cdot F_b)_{w_{j+1}} \tag{4.13}$$

for $j \geq 0$. And since $\widetilde{N}$ meets F_a and F_b transversely at u_0 and w_0 if it meets the curves at these points, we have

$$\mu_0 \leq \mu \leq \mu_0 + 1 \text{ and } \mu_{-1} \leq \mu \leq \mu_{-1} + 1 \tag{4.14}$$

where $\mu = \mu_0 + 1$ iff $w_0 \in \widetilde{N}$ and $\mu = \mu_{-1} + 1$ iff $u_0 \in \widetilde{N}$. Hence

$$(\widetilde{\Gamma} \cdot F_a)_{u_0} = (\widetilde{\Gamma} \cdot F_b)_{w_0} = \mu. \tag{4.15}$$

Now we are ready to estimate the total δ-invariant $\delta(Y_t^{(\alpha)}, \Gamma)$ of $Y_t^{(\alpha)}$ in the neighborhood of Γ. First, in the neighborhood of the rational double point p_α where $Y^{(\alpha)}$ is totally separated by Proposition 4.2, we may apply Corollary 3.8 to conclude (noticing (4.15))

$$\delta(Y_t^{(\alpha)}, p_\alpha) \geq \mu^2. \tag{4.16}$$

Second, in the neighborhood of each point $s = N \cap \nu(\overline{w_i u_{i+1}})$ with $s \notin \{\nu(w_i), \nu(u_{i+1})\}$, $Y^{(\alpha)}$ is totally separated by Proposition 4.2 and hence Lemma 3.6 can be applied (see also Remark 3.7). It follows that

$$\delta(Y_t^{(\alpha)}, s) \geq \mu_i. \tag{4.17}$$

Let $s_i = (N \cap \nu(\overline{w_i u_{i+1}})) \backslash \{\nu(w_i), \nu(u_{i+1})\}$. Obviously, $s_i = \emptyset$ if either $w_i \in \widetilde{N}$ or $u_{i+1} \in \widetilde{N}$. By (4.14), $s_0 = \emptyset$ iff $\mu = \mu_0 + 1$. Therefore,

$$\delta(Y_t^{(\alpha)}, s_0) \geq (\mu_0 + 1 - \mu)\mu_0 \tag{4.18}$$

by (4.17), where we let $\delta(Y_t^{(\alpha)}, s_i) = 0$ if $s_i = \emptyset$. Similarly,

$$\delta(Y_t^{(\alpha)}, s_{-1}) \geq (\mu_{-1} + 1 - \mu)\mu_{-1}. \tag{4.19}$$

Let $0 \leq a_0 < a_1 < a_2 < ... < a_n < ...$ be the sequence of integers such that

$$\begin{aligned} \mu_0 = ... = \mu_{a_0} &> \mu_{a_0+1} = \mu_{a_0+2} = ... = \mu_{a_1} \\ &> \mu_{a_1+1} = \mu_{a_1+2} = ... = \mu_{a_2} \\ &> ... > \mu_{a_{n-1}+1} = \mu_{a_{n-1}+2} = ... = \mu_{a_n} > .... \end{aligned} \tag{4.20}$$

Notice that for $i > 0$, $s_i = \emptyset$ iff $\mu_{i-1} \neq \mu_i$ by (4.13). Therefore,

$$\sum_{i>0} \delta(Y_t^{(\alpha)}, s_i) \geq a_0\mu_0 + \sum_{i>0}(a_i - a_{i-1} - 1)\mu_{a_i} \tag{4.21}$$

by (4.17). Notice that

$$\sum_{i\geq 0} \mu_i = (a_0 + 1)\mu_0 + \sum_{i>0}(a_i - a_{i-1})\mu_{a_i}. \tag{4.22}$$

By (4.21) and (4.22),

$$\begin{aligned} \sum_{i>0} \delta(Y_t^{(\alpha)}, s_i) - \sum_{i\geq 0} \mu_i &= -\sum_{i\geq 0} \mu_{a_i} \\ &\geq -(\mu_0 + (\mu_0 - 1) + ... + 2 + 1) \\ &= -\frac{\mu_0(\mu_0 + 1)}{2}. \end{aligned} \tag{4.23}$$

By the same argument, we have

$$\sum_{i<-1} \delta(Y_t^{(\alpha)}, s_i) - \sum_{i<0} \mu_i \geq -\frac{\mu_{-1}(\mu_{-1} + 1)}{2}. \tag{4.24}$$

Putting (4.11), (4.14), (4.16), (4.18), (4.19), (4.23) and (4.24) altogether, we obtain

$$\begin{aligned}\delta(Y_t^{(\alpha)}, \Gamma) &\geq \delta(Y_t^{(\alpha)}, p_\alpha) + \delta(Y_t^{(\alpha)}, s_0) + \delta(Y_t^{(\alpha)}, s_{-1}) \\ &\quad + \sum_{i>0} \delta(Y_t^{(\alpha)}, s_i) + \sum_{i<-1} \delta(Y_t^{(\alpha)}, s_i) \\ &\geq m + \frac{1}{2}(\mu - \mu_0)^2 + \frac{1}{2}(\mu - \mu_{-1})^2 \\ &\quad - \frac{1}{2}(\mu - \mu_0) - \frac{1}{2}(\mu - \mu_{-1}) = m.\end{aligned} \tag{4.25}$$

This finishes the proof of (2.35) and hence the first part of Proposition 2.9.

It remains to find out what happens if $\delta(Y_t^{(\alpha)}, \Gamma) = m$.

Proposition 4.3. *Suppose that $\delta(Y_t^{(\alpha)}, \Gamma) = m$. Then*

- A1. *all the singularities of $Y_t^{(\alpha)}$ in the neighborhood of Γ actually lie in the neighborhoods of the points p_α and s_i;*
- A2. *the equality holds in* (4.16)*;*
- A3. *the equality holds in* (4.17) *for each $s = N \cap \nu(\overline{w_i u_{i+1}})$ with $s \notin \{\nu(w_i), \nu(u_{i+1})\}$;*
- A4. *$\widetilde{N}$ meets F_a and F_b transversely at each intersection, or equivalently,*

$$|\mu_i - \mu_{i+1}| \leq 1 \tag{4.26}$$

  for all i; in particular, $\mu_{-k} = \mu_{l-1} = 1$;
- A5. *for $-k \leq i \leq l-1$, each component of $\widetilde{Y}_0$ that dominates $\nu(\overline{w_i u_{i+1}})$ maps birationally to $\nu(\overline{w_i u_{i+1}})$, i.e., there are no multiple covers of $\nu(\overline{w_i u_{i+1}})$ on $\widetilde{Y}_0$.*

Remark 4.4. In summary, the numerical relations among μ and μ_i are given by (4.8), (4.11), (4.14) and (4.26). Those readers interested in the enumerative aspect of this problem may have already noticed that the number of such sequences $\{\mu, \mu_i\}$ can be expressed in terms of partition numbers. As we already know, the partition numbers have to pop up somewhere by the works of Yau-Zaslow [Y-Z] and Bryan-Leung [B-L]. Figure 6 shows the simplest possible S-chain, corresponding to the case that $\mu_i = 1$ for $-k \leq i \leq l-1$.

Proof of Proposition 4.3. Since $\delta(Y_t^{(\alpha)}, \Gamma) = m$, all the equalities in (4.25) must hold. Then (A1), (A2) and (A3) follow immediately.

As for (A4), we notice that the equality in (4.23) has to hold. So we must have $\mu_{a_0} = \mu_0$, $\mu_{a_1} = \mu_0 - 1$, $\mu_{a_2} = \mu_0 - 2$ and so on, where

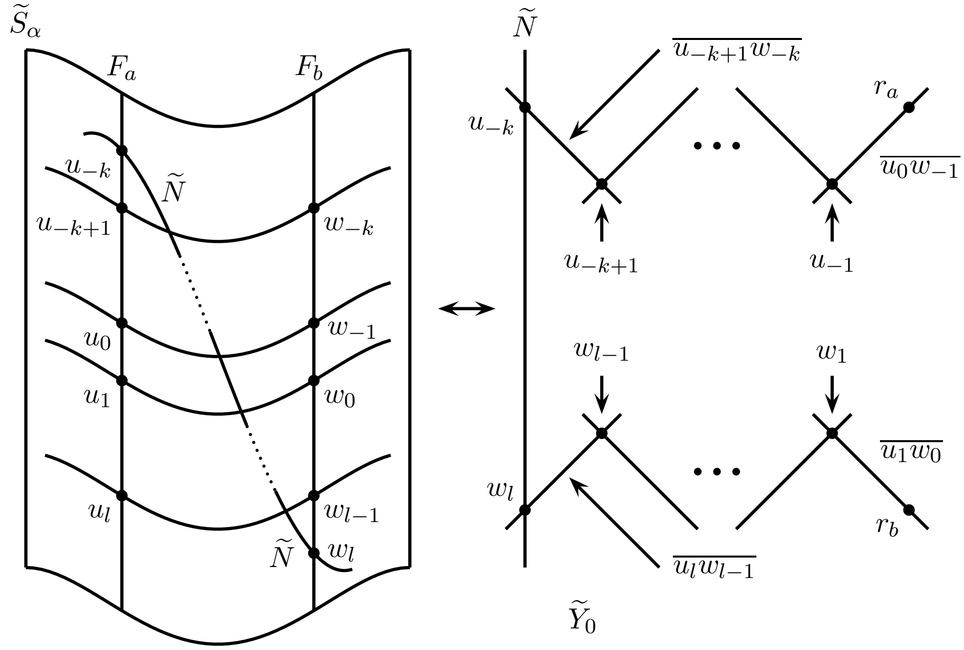

FIGURE 6. An admissible S-chain

$\{a_i\}$ are defined by (4.20). It follows immediately that (4.26) holds for $i \geq 0$. Similarly, (4.26) holds for $i < 0$. And by (4.12) and (4.13), we see that $\widetilde{N}$ meets F_a and F_b transversely everywhere.

Obviously, (A5) holds for $\nu(\overline{w_{l-1}u_l})$ and $\nu(\overline{w_{-k}u_{-k+1}})$ since $\mu_{-k} = \mu_{l-1} = 1$. Suppose that (A5) fails for some $\nu(\overline{w_iu_{i+1}})$ with $i \geq 0$ and i is the largest number with this property. Then there exists a component $M \subset \widetilde{Y}_0$ dominating $\nu(\overline{w_iu_{i+1}})$ with a map of degree at least 2. We claim that

> $(*)$ M is joined to at least two different components $M_1, M_2 \subset \widetilde{Y}_0$ over the point $\nu(u_{i+1})$, where $M_j = \widetilde{N}$ or M_j dominates $\nu(\overline{w_{i+1}u_{i+2}})$ for $j = 1, 2$.

If the map $M \to \nu(\overline{w_iu_{i+1}})$ is not totally ramified over $\nu(u_{i+1})$, there are at least two distinct points $x_1 \neq x_2 \in M$ such that $\pi(x_j) = \nu(u_{i+1})$ for $j = 1, 2$ where $\pi : \widetilde{Y} \to Y^{(\alpha)} \subset X^{(\alpha)}$ is the map from $\widetilde{Y}$ to $Y^{(\alpha)}$. Then by Lemma 3.2, the branch of M at x_j is joined to a component M_j over the point $\nu(u_{i+1})$ for $j = 1, 2$, where $M_j = \widetilde{N}$ or $\pi(M_j) = \nu(\overline{w_{i+1}u_{i+2}})$. This justifies our claim $(*)$ in the case that $\pi : M \to \nu(\overline{w_iu_{i+1}})$ is not totally ramified over $\nu(u_{i+1})$.

If $\pi : M \to \nu(\overline{w_iu_{i+1}})$ is totally ramified over $\nu(u_{i+1})$, $\pi(M)$ meets $F_{p_{\alpha-1}}$ at $\nu(u_{i+1})$ with multiplicity at least 2. Again by Lemma 3.2 (see also Remark 3.4), M is joined to a union of components $\cup M_j$ over

$\nu(u_{i+1})$ such that $\pi(\cup M_j) \subset \nu(\overline{w_{i+1}u_{i+2}}) \cup N$ and $\pi(\cup M_j)$ meets $F_{p_{\alpha-1}}$ at $\nu(u_{i+1})$ with multiplicity at least 2. Our assumption on i implies that (A5) holds for $\nu(\overline{w_{i+1}u_{i+2}})$, i.e., every component of $\widetilde{Y}_0$ dominating $\nu(\overline{w_{i+1}u_{i+2}})$ maps birationally to $\nu(\overline{w_{i+1}u_{i+2}})$. And since $\widetilde{N}$ meets F_b transversely at w_{i+1} if $w_{i+1} \in \widetilde{N}$, we see that $\cup M_j$ contains at least two different components dominating either N or $\nu(\overline{w_{i+1}u_{i+2}})$ and hence $(*)$ follows.

Starting with $(*)$, we may argue as before to show that each M_j is joined by a chain of components over $\nu(\overline{w_{i+2}u_{i+3}}) \cup \nu(\overline{w_{i+3}u_{i+4}}) \cup ... \cup \nu(\overline{w_{l-1}u_l})$ to $\widetilde{N}$ for $j = 1, 2$. And hence there is a circuit in the dual graph of $\widetilde{Y}_0$. Contradiction. So (A5) holds for each $\nu(\overline{w_i u_{i+1}})$ with $i \geq 0$. A similar argument shows that (A5) holds for each $\nu(\overline{w_i u_{i+1}})$ with $i < 0$. □

With Proposition 4.3, the second part of Proposition 2.9 is almost immediate. In the neighborhood of p_α, $Y^{(\alpha)}$ consists of 2μ local irreducible components corresponding to 2μ branches of $\widetilde{Y}_0$ over p_α. And since the equality holds in (4.16), $Y_t^{(\alpha)}$ has exactly μ^2 nodes as singularities in the neighborhood of p_α by Corollary 3.8. In the neighborhood of a point $s = N \cap \nu(\overline{w_i u_{i+1}})$ with $s \notin \{\nu(w_i), \nu(u_{i+1})\}$, $Y^{(\alpha)}$ consists of $\mu_i + 1$ local irreducible components. And since the equality holds in (4.17), $Y_t^{(\alpha)}$ has exactly μ_i nodes as singularities in the neighborhood of s by Lemma 3.6. Therefore, $Y_t^{(\alpha)}$ is nodal if $\delta(Y_t^{(\alpha)}, \Gamma) = m$. This finishes the proof of Proposition 2.9.

Although it is no longer necessary for our purpose, it will be interesting to classify all the possible configurations for the stable reduction $\widetilde{Y}_0$. Actually, this is not hard given everything we have done so far. Next, we will give a description for $\widetilde{Y}_0$ without justification and leave the readers to verify the details.

Let us contract some curves on $\widetilde{Y}_0$ to make $\widetilde{Y} \to Y$ into a stable map. Remember that we start with the stable map $\widetilde{Y} \to Y^{(\alpha)}$.

Among the components of $\widetilde{Y}_0$ that dominate E,

1. there is only one component $\widetilde{N}$ dominating E with a map of degree μ and the rest each map to E birationally;
2. the map $\widetilde{N} \to \widetilde{E}$ is unramified over a and b, where $\widetilde{E}$ is the normalization of E and $a, b \in \widetilde{E}$ are the two points over the node $p \in E$;
3. two components M_1 and M_2 only meet at a point x over the node p; in addition, $M_1 \cup M_2$ maps biholomorphically to E locally at x, i.e., the two branches of M_1 and M_2 at x must map to different

branches of E at p; using the terminology of [B-L], we say that there is a "branch jump" whenever two components meet;

4. for each $x \in \widetilde{N}$ over p, there is a chain of curves $\cup M_i$ attached to $\widetilde{N}$ at p with each M_i dominating E; and each component $M \neq \widetilde{N}$ dominating E lies on one of these 2μ chains;
5. let $\lambda(x)$ be the length of the chain of curves attached to the point $x \in \widetilde{N}$ over p; obviously,

$$\sum \lambda(x) + \mu = m \tag{4.27}$$

where we sum over all the 2μ points $x \in \widetilde{N}$ that map to p;
6. for any two points $x_1 \neq x_2 \in \widetilde{N}$ over a, $\lambda(x_1) \neq \lambda(x_2)$; similarly, for any two points $y_1 \neq y_2 \in \widetilde{N}$ over b, $\lambda(y_1) \neq \lambda(y_2)$;
7. $\widetilde{N}$ meets $\widetilde{C}$ at a point over $q = C \cap E$, where $\widetilde{C}$ is the component of $\widetilde{Y}_0$ dominating C.

Let $x_1, x_2, ..., x_\mu$ be the points of $\widetilde{N}$ over a and $y_1, y_2, ..., y_\mu$ be the points of $\widetilde{N}$ over b. Then $\{x_i\}$ map to the points among $u_{-k}, u_{-k+1}, ..., u_l$ and $\{y_i\}$ map to the points among $w_{-k}, w_{-k+1}, ..., w_l$. Let $\lambda_i = \lambda(x_i)$ and $\lambda_{-i} = \lambda(y_i)$ and we order x_i and y_i such that

$$\lambda_1 > \lambda_2 > ... > \lambda_\mu \geq 0 \tag{4.28}$$

and

$$\lambda_{-1} > \lambda_{-2} > ... > \lambda_{-\mu} \geq 0 \tag{4.29}$$

where $\lambda_\mu = 0$ if and only if x_μ maps to $u_0 = r_a$ and $\lambda_{-\mu} = 0$ if and only if y_μ maps to $w_0 = r_b$. Under these notations, we may rewrite (4.27) as

$$\sum_{i=-\mu}^{\mu} \lambda_i + \mu = m \tag{4.30}$$

where we let $\lambda_0 = 0$. Later in Appendix B when we count the number of rational curves on a K3 surface, we are basically counting the number of the sequences $\{\mu, \lambda_i\}$ satisfying (4.28), (4.29) and (4.30). Figure 7 shows the configuration of $\widetilde{Y}_0$. Also see Figure 6 for the simplest possible configuration of $\widetilde{Y}_0$, corresponding to the case that $\mu = 1$.

It is also worthwhile to point out that $\{\mu, \lambda_i\}$ are uniquely determined by $\{\mu, \mu_j\}$ and vice versa. Actually, we can describe their relation explicitly as follows: the Young tableau of $(\lambda_1, \lambda_2, ..., \lambda_\mu)$ is dual to the Young tableau of $(\mu_{-1}, \mu_{-2}, ..., \mu_{-k})$ and the Young tableau of $(\lambda_{-1}, \lambda_{-2}, ..., \lambda_{-\mu})$ is dual to the Young tableau of $(\mu_0, \mu_1, ..., \mu_{l-1})$ (see Figure 8).

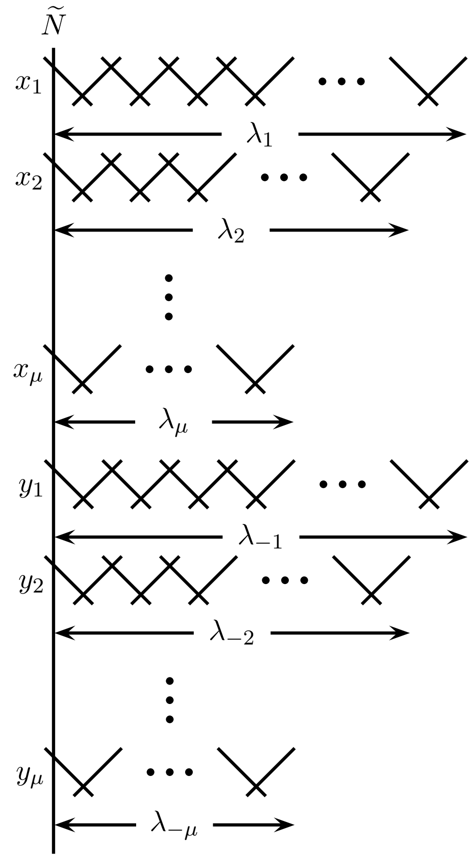

FIGURE 7. $\widetilde{Y}_0$

5. PROOFS OF PROPOSITION 2.7 AND 2.8

5.1. **Proof of Proposition 2.7.** Although the proposition says that we make a base change of a one-size-fits-all degree α at the very beginning, in practice we have no idea of what values α should take before we start to blow up X and Y. So our proof goes as follows: we start with an α for which the proposition might fail, then we make a sequence of base changes depending on where it fails and finally we will obtain an α such that everything in the proposition holds.

Suppose that the proposition holds for $Y^{(n)}$ and $Y_0^{(n)}$ contains E_n with multiplicity μ. So we start with $n = 0$ and $\mu = m$ and we will show that eventually either $n = \alpha$ or $\mu = 0, 1$.

Suppose that $\mu \geq 2$. Pick an arbitrary smooth point $b \neq q_n \in E_n$ of E_n. Locally at b, the curve $Y_0^{(n)}$ is given by $z^\mu = 0$ in Δ^2_{wz}. With a suitable choice of the coordinate z, the family $Y^{(n)}$ is locally given by

$$z^\mu + t^{a_1} f_1(t,w) z^{\mu-2} + t^{a_2} f_2(t,w) z^{\mu-3} + ... + t^{a_{\mu-1}} f_{\mu-1}(t,w) = 0 \tag{5.1}$$

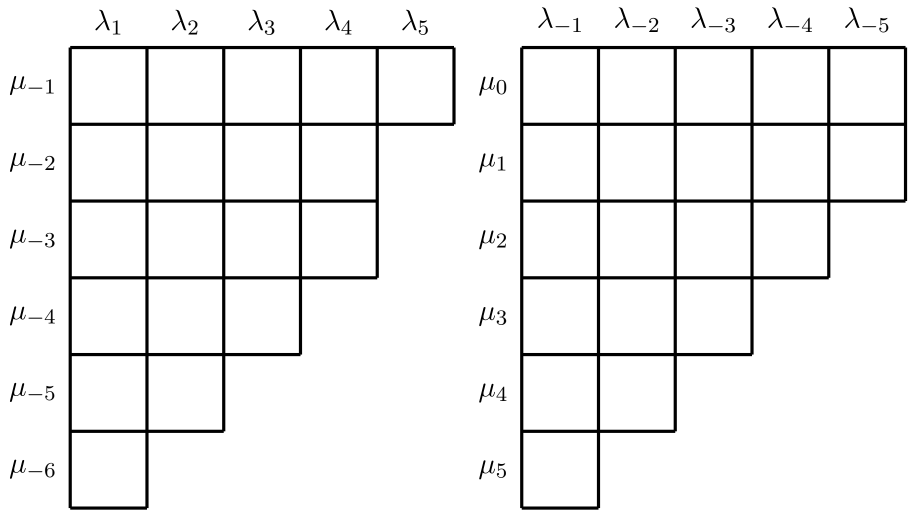

FIGURE 8. Relation between $\{\lambda_i\}$ and $\{\mu_j\}$

in Δ^3_{wzt}, where $a_i > 0$ and $f_i(0,0) \neq 0$ for $i = 1, 2, ..., \mu - 1$. Let

$$\beta = \min_{1 \le i \le \mu - 1} \frac{a_i}{i+1}. \tag{5.2}$$

A base change might be needed in order to make β into a positive integer and we have to modify the sequence (2.30) after a base change. But the bottom line is that μ does not change in the process. So let us assume that β is a positive integer.

If $\beta > 1$, the local equation (5.1) shows that $M = Y^{(n+1)} \cap S_{n+1}$ contains a section of $S_{n+1} \to E_n$ with multiplicity μ and $E_n \not\subset M$. And local computations of $Y^{(n)}$ at p_n and q_n show that M meets E_n only at q_n and it meets E_n at q_n transversely. So $M = F_{q_n} \cup \mu G$, where $G \in |\mathcal{O}_{S_{n+1}}(E_n)|$ and $G \neq E_n$. If $G \neq E_{n+1}$, we are done with the proof of Proposition 2.7 since any further blowups of $Y^{(n+1)}$ will only produce more F_{q_i}'s, i.e., $Y^{(n+k)} \cap S_{n+k}$ will consist only of $F_{q_{n+k-1}}$ for $k > 1$.

So we can apply this argument to every $1 \le k \le \beta - 1$: either $Y^{(n+k)} \cap S_{n+k} = F_{q_{n+k-1}} \cup \mu E_{n+k}$ for each k or this fails for certain k such that $Y^{(n+k)} \cap S_{n+k} = F_{q_{n+k-1}} \cup \mu G$ with $G \neq E_{n+k}$, in which case we are done.

Let us assume that $Y^{(n+k)} \cap S_{n+k} = F_{q_{n+k-1}} \cup \mu E_{n+k}$ for each $k = 1, 2, ..., \beta - 1$.

Due to our choice (5.2) of β, $M = Y^{(n+\beta)} \cap S_{n+\beta}$ consists of at least two components, each of which dominates $E_{n+\beta-1}$ with a map

of degree strictly less than μ, and $E_{n+\beta-1} \not\subset M$. Again, local computations of $Y^{(n+\beta-1)}$ at $p_{n+\beta-1}$ and $q_{n+\beta-1}$ show that M meets $E_{n+\beta-1}$ only at $q_{n+\beta-1}$ and it meets $E_{n+\beta-1}$ at $q_{n+\beta-1}$ transversely. So $M \in |\mathcal{O}_{S_{n+\beta}}(F_{q_{n+\beta-1}} + \mu E_{n+\beta-1})|$. It remains to show that $F_{q_{n+\beta-1}} \subset M$ if $n + \beta < \alpha$.

Let $\nu : \widetilde{E}_{n+\beta-1} \to E_{n+\beta-1}$ be the normalization of $E_{n+\beta-1}$. It induces the normalization $\nu : \mathbb{P}^1 \times \widetilde{E}_{n+\beta-1} \to S_{n+\beta} \cong \mathbb{P}^1 \times E_{n+\beta-1}$ of $S_{n+\beta}$. Let $a, b \in \widetilde{E}_{n+\beta-1}$ be the preimages of $p_{n+\beta-1}$ and let F_a and F_b be the fibers over a and b.

Let ϕ_{ab} be the natural identification between F_a and F_b as defined in 3.1. We can think of $S_{n+\beta}$ as obtained from $\mathbb{P}^1 \times \widetilde{E}_{n+\beta-1}$ by gluing F_a and F_b via ϕ_{ab}. Let $r_a \in F_a$ and $r_b \in F_b$ be the preimages of the rational double point $p_{n+\beta}$ of $X^{(n+\beta)}$. Obviously, $\phi_{ab}(r_a) = r_b$.

Let $\widetilde{M} = \nu^{-1}(M)$. If $\widetilde{M}$ meets F_a at a point $s_a \neq r_a$ with multiplicity k, the branches of $\widetilde{M}$ at s_a will map to the branches of M lying on one of two surfaces of $X_0^{(n+\beta)}$ at $\nu(s_a)$. Recall that $X^{(n+\beta)}$ is locally given by $xy = t^{n+\beta}$ at $\nu(s_a)$. So we can apply Lemma 3.2 to conclude that there exist branches of M lying on the other surface of $X_0^{(n+\beta)}$ at $\nu(s_a)$ and the branches on both surfaces meet $F_{p_{n+\beta-1}}$ at $\nu(s_a)$ with the same multiplicity k. Correspondingly, $\widetilde{M}$ must meet F_b at $s_b = \phi_{ab}(s_a)$ with the same multiplicity k. And since $\widetilde{M} \in |F_{q_{n+\beta-1}} + \mu \widetilde{E}_{n+\beta-1}|$, we draw the conclusion that if $\widetilde{M}$ meets F_a at a point $s_a \neq r_a$ with multiplicity k, it must meet F_b at s_b with the same multiplicity k and hence it must contain the curve $\overline{s_a s_b}$ with multiplicity k. Similarly, if $\widetilde{M}$ meets F_b at a point $s_b \neq r_b$ with multiplicity k, $\widetilde{M}$ must meet F_a at $s_a = \phi_{ba}(s_b)$ with the same multiplicity k and hence it must contain the curve $\overline{s_a s_b}$ with multiplicity k. Therefore, if we let $\widetilde{N} \subset \widetilde{M}$ be the irreducible component of $\widetilde{M}$ with $\widetilde{N} \in |F_{q_{n+\beta-1}} + \gamma \widetilde{E}_{n+\beta-1}|$ for some $\gamma \leq \mu$, we see that $\widetilde{N}$ meets F_a and F_b only at r_a and r_b. But then $\overline{r_a r_b} \subset \widetilde{N}$ if $\gamma > 0$. Therefore, $\gamma = 0$ and $F_{q_{n+\beta-1}} \subset M$.

Since M has at least two components which dominates $E_{n+\beta-1}$, the multiplicity μ' of $E_{n+\beta}$ in M is strictly less than μ. Now the proposition holds for $Y^{(n+\beta)}$ and $Y_0^{(n+\beta)}$ contains $E_{n+\beta}$ with multiplicity $\mu' < \mu$. We see that the value of μ has been reduced.

Finally, if $\mu = 0$, there is nothing left to do. If $\mu = 1$, no further base changes are needed; we just have to verify that $F_{q_{n+k-1}} \subset M = Y^{(n+k)} \cap S_{n+k}$ for $1 \leq k \leq \alpha - n - 1$, the argument for which goes exactly as before. This completes the proof of Proposition 2.7.

5.2. **Proof of Proposition 2.8.** Suppose that $Y^{(\alpha)} \cap S_n$ contains a component $M \in |\mathcal{O}_{S_n}(E_{n-1})|$ with multiplicity $\mu > 0$ for some $1 \le n \le \alpha - 1$. Namely, M is a wondering component. Let $u \in M$ be the node of M, where $X^{(\alpha)}$ is locally given by $xy = t^n$.

Let $\widetilde{Y} \to Y^{(\alpha)}$ be the stable reduction of $Y^{(\alpha)}$ after normalization defined as before. Let G be the dual graph of the components of $\widetilde{Y}_0$ that map to M (including the curves contracted to a point on M) and let us remove from G all the vertices of degree 0 or 1 that represent contractible curves. So $\deg([R]) \ge 2$ for any $[R] \in G$ representing a contractible curve R, where we let $[A]$ denote the vertex of G representing the component $A \subset \widetilde{Y}_0$ and $\deg([A])$ denote the degree of $[A]$ in G.

Let $\widetilde{M} \subset \widetilde{Y}_0$ be a component of $\widetilde{Y}_0$ dominating M and let $\widetilde{u} \in \widetilde{M}$ be one of the points over the node u. The branch of $\widetilde{M}$ at $\widetilde{u}$ maps to a branch of M lying on one of two surfaces of $X_0^{(\alpha)}$ at u. So by Lemma 3.2, the branch of $\widetilde{M}$ at $\widetilde{u}$ is joined by a chain of contractible curves to a branch of $\widetilde{Y}_0$ that maps to the branch of M lying on the other surface of $X_0^{(\alpha)}$ at u. This is to say that each $\widetilde{u} \in \widetilde{M}$ over u corresponds to an edge of G from $[\widetilde{M}]$. And since there are at least two points of $\widetilde{M}$ mapping to u, we must have $\deg([\widetilde{M}]) \ge 2$ in G.

So every vertex of G has degree at least 2. This is impossible and hence Proposition 2.8 follows.

Appendix A. Deformations of K3 Surfaces

Here we will give a proof for Lemma 2.3.

Without the loss of generality, let us assume that D is very ample; otherwise, we may simplely replace Y by nY and D by nD for some $n >> 0$. Let $g = \dim |\mathcal{O}_X(Y)| = \dim |\mathcal{O}_S(D)|$ and X can be embedded to $\mathbb{P}^g \times \Delta$ by the complete linear series $|\mathcal{O}_X(Y)|$. Let N_S be the normal bundle of S in $\mathbb{P}^g$ and we have the standard exact sequence

$$0 \longrightarrow T_S \longrightarrow T_{\mathbb{P}^g}|_S \longrightarrow N_S \longrightarrow 0 \tag{A.1}$$

on S. From (A.1), we have the exact sequence

$$H^0(N_S) \longrightarrow H^1(T_S) \xrightarrow{f} H^1(T_{\mathbb{P}^g}|_S). \tag{A.2}$$

The embedding $X \hookrightarrow \mathbb{P}^g \times \Delta$ gives an embedded deformation of S in $\mathbb{P}^g$. Therefore, the Kodaira-Spencer map $\mathrm{ks} : T_{\Delta,0} \to H^1(T_S)$ factors through $H^0(N_S)$. Consequently, $\mathrm{ks}(\partial/\partial t)$ lies in the kernel of the map $f : H^1(T_S) \to H^1(T_{\mathbb{P}^g}|_S)$. Therefore, to prove (2.5), it suffices to show

that

$$\ker f = V. \tag{A.3}$$

The Euler sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_{\mathbb{P}^g}(1)^{\oplus(g+1)}\big|_S \to T_{\mathbb{P}^g}|_S \to 0 \tag{A.4}$$

yields an isomorphism from $H^1(T_{\mathbb{P}^g}|_S)$ to $H^2(\mathcal{O}_S)$ since

$$H^i(\mathcal{O}_{\mathbb{P}^g}(1)|_S) = H^i(\mathcal{O}_S(D)) = 0 \tag{A.5}$$

for $i = 1, 2$ by Kodaira vanishing theorem. So we have

$$H^1(T_S) \xrightarrow{f} H^1(T_{\mathbb{P}^g}|_S) \xrightarrow{\sim} H^2(\mathcal{O}_S). \tag{A.6}$$

Let us consider the dual sequence of (A.6), i.e.,

$$\begin{array}{ccccc} H^1(T_S) & \xrightarrow{f} & H^1(T_{\mathbb{P}^g}|_S) & \xrightarrow{\sim} & H^2(\mathcal{O}_S) \\ \times & & \times & & \times \\ H^1(\Omega_S) & \xleftarrow{f^\vee} & H^1(\Omega_{\mathbb{P}^g}|_S) & \xleftarrow{\sim} & H^0(\mathcal{O}_S). \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & & \mathbb{C} & & \mathbb{C} \end{array} \tag{A.7}$$

Obviously, (A.3) is equivalent to saying that the image of the map $f^\vee : H^1(\Omega_{\mathbb{P}^g}|_S) \to H^1(\Omega_S)$ is spanned by $c_1(D)$. So it suffices to prove that

$$\operatorname{Im} f^\vee = \operatorname{Span}\{c_1(D)\}. \tag{A.8}$$

From the dual Euler sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{\mathbb{P}^g} & \longrightarrow & \mathcal{O}_{\mathbb{P}^g}(-1)^{\oplus(g+1)} & \longrightarrow & \mathcal{O}_{\mathbb{P}^g} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{\mathbb{P}^g}|_S & \longrightarrow & \mathcal{O}_{\mathbb{P}^g}(-1)^{\oplus(g+1)}\big|_S & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array} \tag{A.9}$$

we see that

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^g}) & \xrightarrow{\sim} & H^1(\Omega_{\mathbb{P}^g}) \\ \downarrow\wr & & \downarrow\wr \\ H^0(\mathcal{O}_S) & \xrightarrow{\sim} & H^1(\Omega_{\mathbb{P}^g}|_S). \end{array} \tag{A.10}$$

It is a common knowledge that $H^1(\Omega_{\mathbb{P}^g}) = \mathbb{C}$ is generated by $c_1(\mathcal{O}_{\mathbb{P}^g}(1))$. It is not hard to see that the image of $c_1(\mathcal{O}_{\mathbb{P}^g}(1))$ under the map

$$H^1(\Omega_{\mathbb{P}^g}) \xrightarrow{\sim} H^1(\Omega_{\mathbb{P}^g}|_S) \xrightarrow{f} H^1(\Omega_S) \tag{A.11}$$

is $c_1(D)$. This proves (A.8) and (A.8) $\Rightarrow$ (A.3) $\Rightarrow$ (2.5).

For the second part of the lemma, suppose that S is a K3 surface, D is an ample divisor on S and S is embedded into $\mathbb{P}^g$ by $|\mathcal{O}_S(nD)|$ for some $n > 0$. Observe that (A.3) also implies that f is a surjection and hence $H^1(N_S) = 0$ by the exact sequence

$$H^1(T_S) \xrightarrow{f} H^1(T_{\mathbb{P}^g}|_S) \longrightarrow H^1(N_S) \longrightarrow H^2(T_S) = 0. \tag{A.12}$$

So the embedded deformations of S in $\mathbb{P}^g$ are unobstructed. And since $H^0(N_S)$ surjects onto V, for each $v \in V$, there exists an embedded deformation of $S \subset \mathbb{P}^g$ with Kodaira-Spencer class v, i.e., there exists a smooth family X over Δ and an embedding $\varphi : X \hookrightarrow \mathbb{P}^g \times \Delta$ such that $\varphi(X_0) = S$ and the Kodaira-Spencer class of X is v. Let $W \subset X$ be the pullback of the hyperplane divisor of $\mathbb{P}^g$. It follows from $c_1(W_0)/n = c_1(D) \in H^2(X_0, \mathbb{Z})$ that $c_1(W_t)/n \in H^2(X_t, \mathbb{Z})$ and hence it is a Hodge class. And since

$$\mathrm{Pic}(X_t) \cong H^{1,1}(X_t) \cap H^2(X_t, \mathbb{Z}) \tag{A.13}$$

for K3 surfaces, $W_t/n \in \mathrm{Pic}(X_t)$. In addition, since $\mathrm{Pic}(X_t)$ is torsion free by (A.13), W_t/n is unique in $\mathrm{Pic}(X_t)$. Hence $W \sim_{\mathrm{lin}} nY$ for some divisor $Y \subset X$, where $\sim_{\mathrm{lin}}$ is the linear equivalence. Obviously, $Y_0 \sim_{\mathrm{lin}} D$ and since $h^0(\mathcal{O}_{X_t}(Y_t)) = h^0(\mathcal{O}_{X_0}(Y_0))$, Y can be chosen such that $Y_0 = D$. We are done.

Remark A.1. Let $\mathcal{M}_g$ be the moduli space of K3 surfaces of genus g. Lemma 2.3 says that every connected component of $\mathcal{M}_g$ is smooth of dimension 19. Therefore, we obtain an elementary proof for this well-known result, which was originally proved using transcendental methods. See also [CLM] for another elementary proof of $\dim \mathcal{M}_g = 19$.

On the other hand, it is also known from the transcendental theory of K3 surfaces that $\mathcal{M}_g$ is irreducible. Note that the irreducibility of $\mathcal{M}_g$ is fundamental to our degeneration argument, since we rely on the very fact that every K3 surface can be deformed to a BL K3 surface. However, it does not seem to be any way of avoiding the use of deep transcendental theory in order to assert the irreducibility of $\mathcal{M}_g$.

Appendix B. Recovery of the Counting Formula of Yau-Zaslow-Bryan-Leung

Let N_g be the number of rational curves in the primitive class of a general K3 surface of genus g. We are trying recover the following remarkable formula of Yau-Zaslow [Y-Z] and Bryan-Leung [B-L]:

$$\sum_{g=0}^{\infty} N_g q^g = \frac{q}{\Delta(q)} = \prod_{n=1}^{\infty} (1 - q^n)^{-24} \tag{B.1}$$

where we let $N_0 = 1$ and $N_1 = 24$.

By the analysis in Sec. 4, it is not hard to see the following:

Proposition B.1. *Each possible configuration of the stable reduction $\widetilde{Y}_0$ counts exactly one for N_g.*

The above proposition is not hard to prove but it is quite tedious to write down the whole argument. Basically, by the analysis in Sec. 4, Y_t has exactly m nodes in the neighborhood of E if Y_0 contains E with multiplicity m; these m nodes approach the points $\nu(\overline{w_{i-1}u_i}) \cap N$ and p_α as $t \to 0$. In order to prove Proposition B.1, one just has to show that the points $\nu(\overline{w_{i-1}u_i}) \cap N$ and p_α can be deformed to m nodes on the general fiber in a "unique" way. See e.g. [CH1], [CH2], [CH3] and [C1] for how to carry out this line of argument. We will leave the details to the readers.

So it suffices to count the number of possible configurations of $\widetilde{Y}_0$ according to the description given at the end of Sec. 4. The number of possible configurations of $\widetilde{Y}_0$ over E is the same as the number of the sequences $\{\mu, \lambda_i\}$ satisfying (4.28), (4.29) and (4.30). We claim that

Proposition B.2. *There are exactly $P(m)$ sequences $\{\mu, \lambda_i\}$ satisfying* (4.28)*,* (4.29) *and* (4.30)*, where $P(m)$ is the partition number of m, i.e.,*

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{m=0}^{\infty} P(m) q^m. \tag{B.2}$$

Assume that Proposition B.2 holds and then the total number of possible configurations of $\widetilde{Y}_0$ is

$$\sum_{m_1+m_2+...+m_{24}=g} P(m_1)P(m_2)...P(m_{24}) \tag{B.3}$$

where $m_1, m_2, ..., m_{24}$ are the multiplicities of Y_0 along the 24 rational nodal curves $F_1, F_2, ..., F_{24} \in |F|$. Obviously, the number given by (B.3) is the coefficient of q^g in the power series (B.1), i.e., N_g. So we are done provided we can prove Proposition B.2.

Let

$$G(q,z) = (1+z)\prod_{k=1}^{\infty}\left((1+q^k z)(1+q^k z^{-1})\right). \tag{B.4}$$

We claim that

Lemma B.3. *The number of the sequences $\{\mu, \lambda_i\}$ satisfying* (4.28)*,* (4.29) *and* (4.30) *is the same as the coefficient of q^m in the power series expansion of $G(q,z)$.*

Proof. It follows from the correspondence

$$\begin{aligned}\{\mu, \lambda_i\} \leftrightarrow & (q^{\lambda_1} z)(q^{\lambda_2} z)...(q^{\lambda_\mu} z) \\ & \cdot (q^{\lambda_{-1}+1} z^{-1})(q^{\lambda_{-2}+1} z^{-1})...(q^{\lambda_{-\mu}+1} z^{-1}).\end{aligned} \tag{B.5}$$

□

Let us write

$$G(q, z) = \sum_{d=-\infty}^{\infty} C_d z^d \tag{B.6}$$

where $C_d \in \mathbb{C}[[q]]$. Then by Lemma B.3, Proposition B.2 holds if and only if

$$C_0 = \sum_{m=0}^{\infty} P(m) q^m = \prod_{n=1}^{\infty} (1 - q^n)^{-1}. \tag{B.7}$$

So it remains to verify (B.7). Our strategy is to first calculate $C_{0,n}$ as in

$$G_n(q, z) = (1 + z) \prod_{k=1}^{n} \big((1 + q^k z)(1 + q^k z^{-1})\big) = \sum_{d=-\infty}^{\infty} C_{d,n} z^d. \tag{B.8}$$

and then take the limit $\lim_{n\to\infty} C_{0,n}$ to obtain C_0. As long as $|q| < 1$ and $z \neq 0$, this process makes sense analytically.

Observe that $G_n(q, z)$ satisfies the functional equation

$$(z + q^n) G_n(q, qz) = (1 + q^{n+1} z) G_n(q, z). \tag{B.9}$$

This gives a recursion relation on the coefficients $C_{d,n}$:

$$\begin{aligned} & q^{d-1} C_{d-1,n} + q^{n+d} C_{d,n} = C_{d,n} + q^{n+1} C_{d-1,n} \\ \Leftrightarrow\ & C_{d-1,n} = \frac{1 - q^{n+d}}{q^{d-1}(1 - q^{n-d+2})} C_{d,n}\end{aligned} \tag{B.10}$$

for $-n < d < n+2$. Combining this with $C_{n+1,n} = q^{n(n+1)/2}$, we obtain

$$C_{0,n} = \frac{(1 - q^{2n+1})(1 - q^{2n})...(1 - q^{n+2})}{(1 - q)(1 - q^2)...(1 - q^n)}. \tag{B.11}$$

Obviously, taking the limit $C_0 = \lim_{n\to\infty} C_{0,n}$ yields (B.7). This finishes the proof of Proposition B.2 and hence the recovery of the counting formula (B.1).

Remark B.4. Here we count the sequences $\{\mu, \lambda_i\}$. An alternative way is to count the sequences $\{\mu, \mu_j\}$ satisfying (4.8), (4.11), (4.14) and (4.26). Since $\{\mu, \lambda_i\}$ and $\{\mu, \mu_j\}$ are "dual" to each other (see Figure 8), we may regard this as the *dual counting* of what we did above and it should give the same number $P(m)$. It turns out that the number of

the sequences $\{\mu, \mu_j\}$ satisfying (4.8), (4.11), (4.14) and (4.26) is given by the coefficient of q^m in the expansion of

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)^2(1-q^2)^2...(1-q^k)^2}. \tag{B.12}$$

This leads to the combinatorial identity

$$\begin{aligned} \prod_{n=1}^{\infty}(1-q^n)^{-1} &= \sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)^2(1-q^2)^2...(1-q^k)^2} \\ &= 1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + ... \end{aligned} \tag{B.13}$$

However, we do not know any direct way to derive (B.13). We believe that (B.13) is known. If it is not, it remains an interesting question trying to find a direct proof for it, a proof without resorting to the correspondence between $\{\mu, \lambda_i\}$ and $\{\mu, \mu_j\}$.

Remark B.5. Notice that we did not recover the full formula of Bryan and Leung. They counted the number of not only rational curves but also genus n curves in the primitive class passing through n general points. It is possible to recover their full formula along our line of argument, but some extra work is needed. The basic setup is the following. Let $Y \subset X$ be a family of genus n curves in the primitive class of X_t passing through n fixed points in general position. Let $x_1, x_2, ..., x_n$ be the n fixed points on $X_0 = S$ and let $G_1, G_2, ..., G_n$ be the fibers of $S \to \mathbb{P}^1$ passing through the points $x_1, x_2, ..., x_n$, respectively. Then Y_0 is supported along G_i and F_j, i.e.,

$$Y_0 = \sum_{i=1}^{n} a_i G_i + \sum_{j=1}^{24} m_j F_j. \tag{B.14}$$

We have analyzed the behavior of Y_t in the neighborhood of $E = F_j$ and classified all possible configurations of the stable reduction $\widetilde{Y}_0$ over F_j. However, we have not yet done the same for Y along G_i, which is required for our counting. On the other hand, this can be carried out along the same line of argument as we did for F_j. That is, we will repeatedly blow up X along $G = G_i$ until we obtain a nontrivial ruled surface S_α over G on the central fiber. Then we will analyze the proper transform of Y under the blowups in much the same way as we did in Sec. 4. It will finally come down to the study of certain curves on S_α. Hopefully, we will do this in a future paper.

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